

Rational-slope rigidity with identity holonomy through finite-amplitude resonant cells

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Abstract

We distinguish ordinary and identity-holonomy minimizing foliations for $u'' = -V_u(x, u)$ on $\mathbb{T}^2$. Under the ordinary deck-invariant definition, $V(u) = -\frac{1}{2} \sin^4(\pi u)$ admits minimizing foliations of every rotation number although $V_u \not\equiv 0$; its zero-rotation holonomy is nontrivial. If, for each rational rotation, there exists a minimizing foliation with identity primitive holonomy, we prove $V_u \equiv 0$ by reducing to, and classifying, phase-flat gauge-fixed rational cells of a periodic quadratic Lagrangian. The phase-space completion is obtained directly from order, compactness, circle translation numbers, and area preservation.

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1. Introduction and main results

Let

$$\mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad V \in C^\infty(\mathbb{T}_x \times \mathbb{T}_u),$$

and consider the action

$$(1) \quad \mathcal{A}_V[u; a, b] := \int_a^b \left(\frac{1}{2} |u'(x)|^2 - V(x, u(x)) \right) dx.$$

Its Euler–Lagrange equation is

$$(2) \quad u''(x) = Q(x, u(x)), \quad Q := -V_u.$$

Problem 4 in [1] asks whether the existence of minimizing foliations at every slope forces V to be independent of u . This problem is also discussed in Section 5 of [3]. In one base dimension, we firstly have to note that a rational foliation may have nontrivial holonomy on its leaf space. Namely a rational foliation does not necessarily return each leaf to itself after one primitive period.

We prove two statements under explicitly distinguished foliation conventions. The ordinary all-rotation assertion, with foliation understood as in Definition 2.2, has a counterexample. Under identity primitive holonomy at every rational rotation, rigidity holds.

1.1. The ordinary statement counterexample. A lifted ordered foliation of rational rotation $-m/\ell$ is invariant, as a family, under the primitive deck transformation $(x, u) \mapsto (x + \ell, u - m)$. After the vertical deck action has been normalized, this transformation acts on the leaf parameter by an increasing

degree-one lift

$$\mathfrak{h}_{m,\ell} : \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathfrak{h}_{m,\ell}(t+1) = \mathfrak{h}_{m,\ell}(t) + 1.$$

The exact defining identity is given in Definition 2.3. The foliation has *identity primitive holonomy* when

$$\mathfrak{h}_{m,\ell} = \text{Id}.$$

This condition is invariant under every admissible relabelling of the leaves, because a relabelling only conjugates $\mathfrak{h}_{m,\ell}$.

THEOREM 1.1 (Counterexample under ordinary holonomy). *Set*

$$(3) \quad V_*(u) := -\frac{1}{2} \sin^4(\pi u).$$

Then $(V_*)_u \neq 0$. Nevertheless, for every $\rho \in \mathbb{R}$, the torus $\mathbb{T}^2$ admits an ordinary deck-invariant foliation by graphs of class-A minimizing solutions of

$$u'' = -(V_*)_u(u),$$

all having rotation number ρ . For every nonzero rational rotation, the foliation can be chosen with identity primitive holonomy. The zero-rotation foliation has nontrivial primitive holonomy and therefore cannot be put into identity holonomy by changing the leaf parameter.

This proof occupies all of Section 2. This is an *explicit* counterexample, not just an obstruction to a proof, or something obtained only as a limit. The leaves satisfy a first-order equation and are calibrated by an exact square completion. At zero rotation they are the constant solutions together with the explicit heteroclinics

$$u_{k,s}(x) = k + \frac{1}{2} + \frac{1}{\pi} \arctan(\pi(x-s)).$$

The resulting foliation is the graph foliation of the smooth nonvanishing vector field

$$\partial_x + \sin^2(\pi u) \partial_u$$

on $\mathbb{T}^2$. Its primitive return map is the time-one map of $\dot{u} = \sin^2(\pi u)$, hence it has fixed points and is not the identity.

Consequently, the all-rotation rigidity assertion is false already for $n = 1$ when “foliation is understood in the precise sense of Definition 2.2. The counterexample does not apply to a convention which requires every rational leaf to close in its primitive homology class. Our positive theorem addresses that stronger convention. Lemma 2.7 also shows directly that the counterexample has no alternative identity-holonomy zero-rotation minimizing foliation.

1.2. *The identity-holonomy rigidity theorem.* We now state the extra condition for the positive theorem to be true. For every primitive pair

$$(m, \ell) \in \mathbb{Z} \times \mathbb{N}, \quad \gcd(m, \ell) = 1,$$

we use the sign convention

$$(4) \quad \rho = -\frac{m}{\ell}.$$

The condition $\mathfrak{h}_{m,\ell} = \text{Id}$ is equivalent to saying that every lifted leaf closes under the primitive deck transformation:

$$(5) \quad U_{m,\ell}(x + \ell, t) = U_{m,\ell}(x, t) - m.$$

This is the condition used in the proof. It is stronger than saying only that the deck transformation preserves the foliation as a set.

THEOREM 1.2 (Rational-slope rigidity with identity holonomy). *Let $V \in C^\infty(\mathbb{T}^2)$. Suppose that, for every primitive pair (m, ℓ) , there is an ordinary lifted foliation of rotation $-m/\ell$ by graphs of class-A minimizing solutions of (2), and suppose that its primitive holonomy is the identity:*

$$\mathfrak{h}_{m,\ell} = \text{Id}.$$

No continuity, measurability, or differentiability in the rotation parameter is assumed. No transverse differentiability of an individual foliation is assumed. Then

$$V_u \equiv 0.$$

The transverse regularity needed in the proof is obtained from the ordered family itself. After the canonical normalization $U_{m,\ell}(0, t) = t$, each identity-holonomy rational foliation is smooth in (x, t) and satisfies $U_{m,\ell,t} > 0$; in Appendix B. The identity-holonomy assumption is essential in that argument because it makes the positive finite differences periodic in x .

These are two uses of the extra condition:

- (i) The equality $\mathfrak{h}_{0,1} = \text{Id}$ gives

$$U_0(x + 1, c) = U_0(x, c).$$

Therefore $a(x, c) = U_{0,c}(x, c)^2$ is a function on the product torus $\mathbb{T}_x \times \mathbb{T}_c$. Without this equality, a satisfies a mapping-torus deck law instead.

- (ii) For a nonzero rational class, $\mathfrak{h}_{m,\ell} = \text{Id}$ gives (5). Hence each transformed leaf belongs to the closed winding class

$$c(x + \ell) = c(x) - m,$$

and can be used in the gauge-fixed resonant cell problem. Without this equality, the primitive deck transformation moves the leaf to a different leaf and there is no closed cell curve at that phase.

After the reduced rational cells have been proved flat, no holonomy assumption appears again.

1.3. *The reduced finite-amplitude theorem.* The rigidity theorem is reduced to an independent classification for

$$(6) \quad L_a(x, c, v) := \frac{1}{2}a(x, c)v^2, \quad a \in C^\infty(\mathbb{T}_x \times \mathbb{T}_c), \quad a > 0.$$

For a primitive pair (m, ℓ) , a curve in the rational winding class satisfies

$$(7) \quad c(x + \ell) = c(x) - m.$$

The resonant phase

$$(8) \quad Z_{m,\ell}[c](x) := mx + \ell c(x)$$

is ℓ -periodic. Its mean is

$$(9) \quad \Pi_{m,\ell}(c) := \frac{1}{\ell} \int_0^\ell (mx + \ell c(x)) dx \pmod{\ell}.$$

We write $\mathbb{T}_\ell = \mathbb{R}/\ell\mathbb{Z}$. The gauge-fixed cell energy is

$$(10) \quad E_{m,\ell}(Y) := \inf \left\{ \frac{1}{2\ell} \int_0^\ell a(x, c(x)) c'(x)^2 dx : \right. \\ \left. c(x + \ell) = c(x) - m, \quad \Pi_{m,\ell}(c) = Y \right\}, \quad Y \in \mathbb{T}_\ell.$$

If the constant corrector is left free, it absorbs Y and the corresponding value function is flat for every coefficient. Section 4 fixes this gauge explicitly.

Definition 1.3 (Flat rational cell). The (m, ℓ) cell is *flat* if $E_{m,\ell}(Y)$ is independent of $Y \in \mathbb{T}_\ell$. The coefficient a has *flat rational cells* if every primitive cell is flat.

THEOREM 1.4 (Finite-amplitude resonant cell rigidity). *Let $a \in C^\infty(\mathbb{T}^2)$ be positive. The following are equivalent.*

- (I) *Every primitive gauge-fixed rational cell is flat.*
- (II) *There are smooth positive periodic functions A and B such that*

$$a(x, c) = A(x)B(c).$$

- (III) *There are smooth periodic functions X and C such that*

$$\log a(x, c) = X(x) + C(c).$$

- (IV) *One has*

$$(\log a)_{xc} \equiv 0.$$

1.4. *Proof architecture.* The proof of Theorem 1.4 follows the chain

$$\begin{aligned}
 & \text{all gauge-fixed rational cells are flat} \\
 & \implies \text{one ordered rational minimizing foliation for every } \rho \in \mathbb{Q} \\
 & \implies \text{closed calibrations and class-A rational leaves} \\
 & \implies \text{a complete ordered phase-space foliation by minimizers} \\
 (11) \quad & \implies \text{no conjugate points for } L_a \\
 & \implies \text{no conjugate points for } \frac{1}{2}(u_y - B(y, u))^2 \\
 & \implies (B_y + BB_u)_u = 0 \implies B_u = 0 \\
 & \implies a(x, c) = A(x)B(c).
 \end{aligned}$$

The calculations behind each arrow are in the main text whereas the points which are mostly functional-analytic—fixed-endpoint versus compactly supported competitors, Sobolev splicing, weak Stokes, the measurable Green field, and periodic Sturm–Liouville facts—are in the appendices.

The direction $a = A(x)B(c) \Rightarrow$ flat cells is explicit. Put

$$h(c) = \int_0^c \sqrt{B(s)} ds.$$

Then $B(c)c'^2 = (h(c))'^2$, and the weighted Dirichlet minimizers can be written in closed form. Their energy does not depend on the vertical phase.

For the converse, flatness first removes the phase multiplier. The resulting full-class minimizers are ordered by the one-dimensional crossing argument. Their momentum graph carries the closed one-form

$$\lambda = P dc - H(x, c, P) dx, \quad H(x, c, p) = \frac{p^2}{2a(x, c)}.$$

Fenchel's identity then shows that every finite segment is minimizing between its endpoints. Rational minimizing graphs are completed to all rotations by a direct order and compactness argument. Rational gaps are excluded by the calibrated quadratic comparison; irrational gaps are excluded by translation-number monotonicity and finite-area preservation, including the case in which limiting graphs touch. An explicit energy estimate provides properness. Once every phase point lies on a minimizing graph, every extremal is minimizing and the flow has no conjugate points.

The normal form is exact. Define

$$I(x) = \int_0^1 \sqrt{a(x, s)} ds, \quad F(x, c) = \frac{1}{I(x)} \int_0^c \sqrt{a(x, s)} ds,$$

and then choose $y = Y(x)$ with

$$Y_x = \frac{I(x)^{-2}}{\int_0^1 I(t)^{-2} dt}.$$

If $u = F(x, c)$, the action becomes a positive constant multiple of

$$\int \frac{1}{2} (u_y - B(y, u))^2 dy.$$

Its Euler equation is

$$u_{yy} = Q_B(y, u), \quad Q_B = B_y + BB_u,$$

and the Jacobi curvature $K = (Q_B)_u$ is independent of the velocity. The Green–Hopf slab argument on $\mathbb{T}^2 \times \mathbb{R}_v$ gives $K = 0$, after which the characteristic equation for $w = B_u$ is $w' = -w^2$. Smooth periodicity forces $w = 0$.

1.5. *Return to the original potential.* Under identity zero holonomy, the zero foliation can be written as $U_0(x, c)$ with

$$U_0(x+1, c) = U_0(x, c), \quad U_0(x, c+1) = U_0(x, c) + 1, \quad U_{0,c} > 0.$$

Writing an arbitrary graph as $u(x) = U_0(x, c(x))$ gives

$$(12) \quad \left(\frac{1}{2} u'^2 - V(x, u) \right) dx = \omega_0|_{\text{graph } u} + \frac{1}{2} U_{0,c}(x, c)^2 c'^2 dx,$$

where ω_0 is closed. Thus the reduced coefficient is

$$(13) \quad a(x, c) = U_{0,c}(x, c)^2.$$

The other identity-holonomy rational foliations make every gauge-fixed cell of this coefficient flat. Theorem 1.4 gives

$$U_{0,c}(x, c)^2 = A(x)B(c).$$

The vertical deck identity forces the x factor of $U_{0,c}$ to be constant, so

$$U_0(x, c) = \psi(x) + H(c).$$

Substitution in the zero-leaf equation shows that $V_u(x, u)$ is independent of u . Its vertical mean is zero, hence $V_u = 0$.

1.6. *Organization.* Section 2 gives the ordinary definition, the holonomy map, and importantly the complete counterexample. Section 3 states the identity-holonomy hypothesis precisely and carries out the original-to-quadratic reduction. Sections 4 and 5 develop the gauge-fixed cell problem and the separable direction. Sections 6 and 7 construct and calibrate rational foliations. Sections 8 and 9 complete phase space and exclude conjugate points. Sections 10 and 11 prove the normal form and Hopf rigidity. Section 12 assembles the finite-amplitude theorem, and finally Section 13 proves the identity-holonomy rigidity

theorem. The appendices supply the analytic, variational, and circle-map details used in these arguments.

2. Ordinary foliations, holonomy, and the one-dimensional counterexample

We firstly fix the variational meaning of “minimal”, then separate ordinary deck invariance from identity holonomy. This separation is the point of the counterexample, after which there is no contradiction between the negative result in this section and the positive theorem proved later.

2.1. Class-A minimizers and rotation number.

Definition 2.1 (Class-A minimizer). A function $u \in W_{\text{loc}}^{1,2}(\mathbb{R})$ is a *class-A minimizer* of (1) if, for every bounded interval (a, b) and every $\varphi \in W_0^{1,2}(a, b)$,

$$(14) \quad \mathcal{A}_V[u; a, b] \leq \mathcal{A}_V[u + \varphi; a, b].$$

In one dimension, a $W^{1,2}$ function has an absolutely continuous representative and therefore has well-defined endpoint values. Hence Definition 2.1 is equivalent to minimizing on every compact interval against every $W^{1,2}$ competitor with the same endpoints. It is also equivalent to testing only smooth compactly supported variations. We defer those approximation statements to Appendix A, so that the main calculation below can be done directly with fixed endpoints.

A lifted graph $u : \mathbb{R} \rightarrow \mathbb{R}$ has rotation number ρ if

$$(15) \quad \lim_{T \rightarrow +\infty} \frac{u(x+T) - u(x)}{T} = \rho.$$

For every graph used below we have the stronger bounded-deviation property

$$(16) \quad \sup_{x \in \mathbb{R}} |u(x) - \rho x| < \infty,$$

which immediately implies (15).

2.2. Ordinary deck-invariant foliations.

Definition 2.2 (Ordinary lifted minimizing foliation). An *ordinary lifted minimizing foliation of rotation ρ* is a continuous map

$$U : \mathbb{R}_x \times \mathbb{R}_t \longrightarrow \mathbb{R}$$

with the following properties.

- (i) For every t , the graph $x \mapsto U(x, t)$ is a smooth class-A minimizing solution of (2) with rotation number ρ .
- (ii) The vertical deck action is normalized by

$$(17) \quad U(x, t+1) = U(x, t) + 1.$$

- (iii) For every fixed x , the map $t \mapsto U(x, t)$ is a strictly increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$.
- (iv) The family of leaves is invariant under the deck group of $\mathbb{R}^2 \rightarrow \mathbb{T}^2$.

The fourth condition means that an integer translate of a leaf is another leaf of the same family. It does not mean that each leaf is itself fixed by the deck translation corresponding to its rational direction. Definition 2.2 fixes the precise convention used here. For the classical minimal-solution setting and its order structure, see [7] and [2]; no identification of all possible foliation conventions is required for our counterexample.

Now, if we fix a primitive rational rotation

$$\rho = -\frac{m}{\ell}, \quad (m, \ell) \in \mathbb{Z} \times \mathbb{N}, \quad \gcd(m, \ell) = 1.$$

The primitive deck transformation tangent to this homology class is

$$(x, u) \mapsto (x + \ell, u - m).$$

Because the family is invariant and the leaf through each point is unique, this deck transformation induces a unique map on the leaf parameter.

Definition 2.3 (Primitive holonomy). The *primitive holonomy* of an ordinary rational foliation is the unique increasing homeomorphism

$$\mathfrak{h}_{m,\ell} : \mathbb{R} \longrightarrow \mathbb{R}$$

satisfying

$$(18) \quad U(x + \ell, t) = U(x, \mathfrak{h}_{m,\ell}(t)) - m \quad \text{for every } (x, t) \in \mathbb{R}^2.$$

The vertical and primitive deck transformations commute. Using (17) in (18) gives

$$(19) \quad \mathfrak{h}_{m,\ell}(t + 1) = \mathfrak{h}_{m,\ell}(t) + 1.$$

Hence $\mathfrak{h}_{m,\ell}$ is a degree-one lift of an orientation-preserving circle homeomorphism.

Definition 2.4 (Identity primitive holonomy). The rational foliation has *identity primitive holonomy* if

$$\mathfrak{h}_{m,\ell} = \text{Id}.$$

Equivalently,

$$(20) \quad U(x + \ell, t) = U(x, t) - m \quad \text{for every } (x, t).$$

We need one elementary invariance observation. Suppose the same leaves are relabelled by an increasing degree-one lift ψ and write

$$\widetilde{U}(x, s) = U(x, \psi(s)).$$

Then the new holonomy is

$$(21) \quad \tilde{\mathfrak{h}}_{m,\ell} = \psi^{-1} \circ \mathfrak{h}_{m,\ell} \circ \psi.$$

In particular, a nonidentity holonomy cannot be made equal to the identity by normalizing the phase. This is why the extra hypothesis in Theorem 1.2 is geometric.

2.3. *The counterexample potential.* Set

$$(22) \quad V_*(u) := -\frac{1}{2} \sin^4(\pi u)$$

and define

$$(23) \quad b(u) := \sin^2(\pi u).$$

Then

$$-V_*(u) = \frac{1}{2} b(u)^2$$

and

$$(24) \quad (V_*)_u(u) = -2\pi \sin^3(\pi u) \cos(\pi u),$$

so $(V_*)_u \not\equiv 0$.

The action is the following

$$(25) \quad \mathcal{A}_{V_*}[u; a, b] = \int_a^b \left(\frac{1}{2} u'^2 + \frac{1}{2} b(u)^2 \right) dx.$$

The potential and the first-order fields used below are shown in Figure 1.

2.4. *The exact calibration identity.* For $E \geq 0$, define

$$(26) \quad g_E(u) := \sqrt{b(u)^2 + 2E}.$$

At $E = 0$ this means $g_0 = b$, because $b \geq 0$. Fix a sign $\sigma \in \{-1, +1\}$. We have the following pointwise algebraic identity for every $(u, v) \in \mathbb{R}^2$:

$$(27) \quad \frac{1}{2} v^2 + \frac{1}{2} b(u)^2 = \frac{1}{2} (v - \sigma g_E(u))^2 + \sigma g_E(u) v - E.$$

Using only $\sigma^2 = 1$ and $g_E(u)^2 = b(u)^2 + 2E$, this is simply the expansion

$$\frac{1}{2} (v - \sigma g_E)^2 + \sigma g_E v - E = \frac{1}{2} v^2 + \frac{1}{2} g_E^2 - E = \frac{1}{2} v^2 + \frac{1}{2} b^2.$$

Let

$$(28) \quad G_E(s) := \int_0^s g_E(r) dr.$$

If $w \in W^{1,2}(a, b)$, the Sobolev chain rule gives

$$(29) \quad g_E(w) w' = (G_E(w))' \quad \text{almost everywhere.}$$

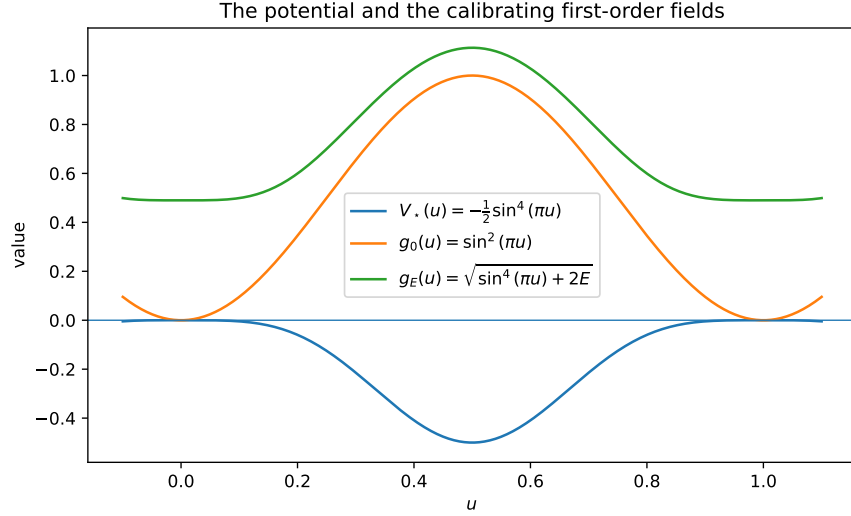


Figure 1. The counterexample potential, the zero-energy field $g_0 = b$, and a positive-energy field g_E . The vanishing of g_0 at the integers is what creates the heteroclinic zero-rotation foliation and its nontrivial holonomy. The Python plotting code was developed with assistance from Codex; see the AI use statement.

Integrating (27) with $v = w'$ therefore gives

$$(30) \quad \mathcal{A}_{V_*}[w; a, b] = \frac{1}{2} \int_a^b (w' - \sigma g_E(w))^2 dx + \sigma (G_E(w(b)) - G_E(w(a))) - E(b - a).$$

The last two terms depend only on the interval and the endpoint values of w .

PROPOSITION 2.5 (Calibration of the first-order branches). *Let u be an entire solution of*

$$(31) \quad u' = \sigma g_E(u).$$

Then u is a class-A minimizer of (25). More precisely, for every compact interval $[a, b]$ and every $w \in W^{1,2}(a, b)$ with $w(a) = u(a)$ and $w(b) = u(b)$,

$$(32) \quad \mathcal{A}_{V_*}[w; a, b] - \mathcal{A}_{V_*}[u; a, b] = \frac{1}{2} \int_a^b (w' - \sigma g_E(w))^2 dx \geq 0.$$

Proof. Apply (30) to w and to u . Their endpoint terms agree. The square for u vanishes by (31). This gives (32). The equivalence with compactly supported competitors is Appendix A. $\square$

Importantly, the same first-order equation also implies the second-order Euler–Lagrange equation. Indeed,

$$(33) \quad \begin{aligned} u'' &= \sigma g'_E(u)u' = g'_E(u)g_E(u) \\ &= b'(u)b(u) = -(V_*)_u(u), \end{aligned}$$

because differentiation of $g_E^2 = b^2 + 2E$ gives $g_E g'_E = bb'$. Hence every calibrated branch is both minimizing and a classical solution. The differential-form interpretation of the endpoint term is recorded in Appendix E; the square calculation above is all that is needed in the main proof.

2.5. Every nonzero rotation number. For $E > 0$, the field g_E is strictly positive. A solution of $u' = g_E(u)$ takes the time

$$(34) \quad T(E) := \int_0^1 \frac{ds}{g_E(s)} = \int_0^1 \frac{ds}{\sqrt{\sin^4(\pi s) + 2E}}$$

to increase by one vertical period. We now check that this time realizes all nonzero slopes.

LEMMA 2.6 (The energy–rotation correspondence). *The map $T : (0, \infty) \rightarrow (0, \infty)$ is smooth, strictly decreasing, and satisfies*

$$\lim_{E \downarrow 0} T(E) = +\infty, \quad \lim_{E \rightarrow \infty} T(E) = 0.$$

Consequently

$$\rho(E) := \frac{1}{T(E)}$$

is a smooth increasing bijection from $(0, \infty)$ onto $(0, \infty)$.

Proof. For $E > 0$, differentiation under the integral sign is legitimate and gives

$$(35) \quad T'(E) = - \int_0^1 \frac{ds}{(b(s)^2 + 2E)^{3/2}} < 0.$$

As $E \downarrow 0$, the integrands increase pointwise to $1/b(s)$. Near $s = 0$,

$$b(s) = \sin^2(\pi s) \sim \pi^2 s^2,$$

so $1/b(s)$ is not integrable. Monotone convergence gives $T(E) \rightarrow +\infty$. Finally,

$$0 < T(E) \leq \frac{1}{\sqrt{2E}},$$

so $T(E) \rightarrow 0$ as $E \rightarrow \infty$. □

Figure 2 shows this monotone correspondence.

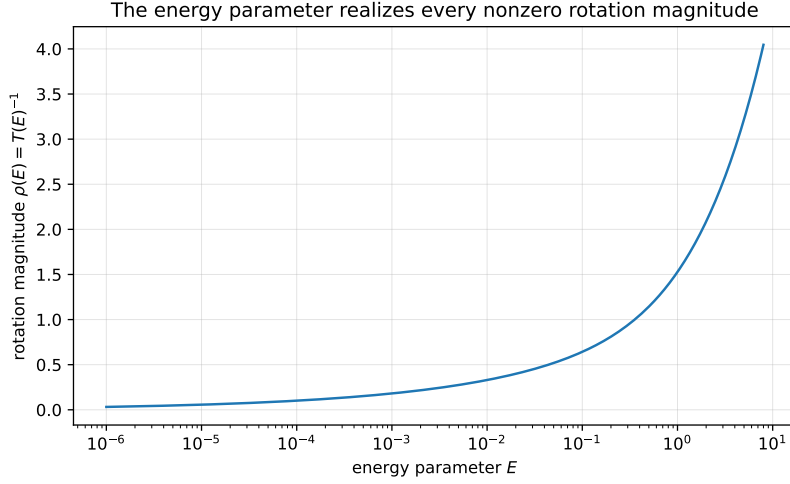


Figure 2. The nonzero rotation magnitude is $\rho(E) = T(E)^{-1}$. The limits at 0 and $+\infty$ prove that every positive magnitude occurs exactly once. The Python plotting code was developed with assistance from Codex; see the AI use statement.

Fix now $\rho \neq 0$, and let $E = E(|\rho|)$ be the unique value satisfying

$$(36) \quad T(E) = \frac{1}{|\rho|}.$$

Define

$$(37) \quad \vartheta_E(u) := \frac{1}{T(E)} \int_0^u \frac{ds}{g_E(s)}.$$

Since g_E is positive and 1-periodic,

$$(38) \quad \vartheta_E(u+1) = \vartheta_E(u) + 1.$$

Hence ϑ_E is an increasing degree-one diffeomorphism of the lifted circle. Let $f_E = \vartheta_E^{-1}$. Then

$$(39) \quad f_E(s+1) = f_E(s) + 1, \quad f'_E(s) = T(E)g_E(f_E(s)).$$

Define

$$(40) \quad U_\rho(x, t) := f_E(t + \rho x).$$

Then

$$(41) \quad \begin{aligned} \partial_t U_\rho &= f'_E(t + \rho x) > 0, \\ \partial_x U_\rho &= \rho T(E)g_E(U_\rho) = \text{sgn}(\rho)g_E(U_\rho). \end{aligned}$$

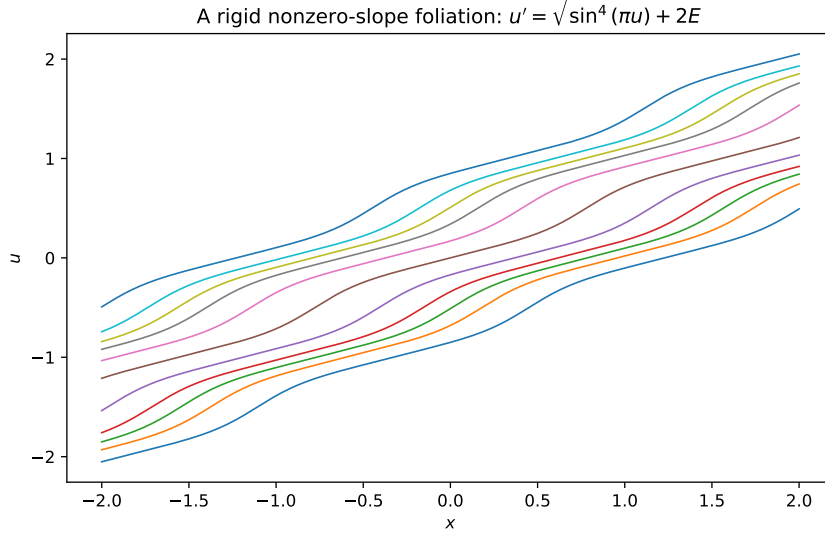


Figure 3. A positive-energy foliation. The leaves never meet, fill the lifted configuration plane, and satisfy the rigid deck law. The negative branch is obtained by reversing the sign of the first-order field. The Python plotting code was developed with assistance from Codex; see the AI use statement.

Therefore every leaf is a calibrated class-A minimizer by Proposition 2.5. Moreover,

$$(42) \quad \begin{aligned} U_\rho(x, t+1) &= U_\rho(x, t) + 1, \\ U_\rho(x+j, t) &= U_\rho(x, t + \rho j) \quad (j \in \mathbb{Z}). \end{aligned}$$

Hence the leaves form an ordinary deck-invariant foliation; in fact, the deck action is rigid for every nonzero ρ .

Because $f_E(s) - s$ is 1-periodic, there is a bounded periodic function ψ_E with $f_E(s) = s + \psi_E(s)$. Consequently

$$(43) \quad U_\rho(x, t) - \rho x = t + \psi_E(t + \rho x)$$

is bounded in x . Every leaf has rotation number ρ .

If $\rho = -m/\ell$ is primitive, (42) gives

$$U_\rho(x + \ell, t) = U_\rho(x, t - m) = U_\rho(x, t) - m.$$

Comparing with (18), we obtain

$$(44) \quad \mathfrak{h}_{m,\ell} = \text{Id} \quad (\rho \neq 0).$$

One of the nonzero-rotation foliations in the counterexample is shown in Figure 3.

2.6. *The zero-rotation foliation.* At $E = 0$, consider the complete scalar flow

$$(45) \quad \dot{y} = b(y) = \sin^2(\pi y).$$

Let $\Phi_s(t)$ denote its time- s map. The vector field is smooth and bounded, so the flow exists for every $s \in \mathbb{R}$. Its periodicity gives

$$(46) \quad \Phi_s(t+1) = \Phi_s(t) + 1,$$

and the flow law gives

$$(47) \quad \Phi_{s+r} = \Phi_s \circ \Phi_r.$$

Differentiation with respect to the initial value yields

$$(48) \quad \partial_t \Phi_s(t) = \exp \left(\int_0^s b'(\Phi_\sigma(t)) d\sigma \right) > 0.$$

Hence every time map is an orientation-preserving diffeomorphism of the lifted circle.

We define

$$(49) \quad U_0(x, t) := \Phi_x(t).$$

For fixed x , the map $t \mapsto U_0(x, t)$ is an increasing diffeomorphism, and

$$(50) \quad \partial_x U_0 = b(U_0), \quad \partial_t U_0 > 0.$$

Every leaf satisfies the zero-energy first-order equation and is therefore a class-A minimizer. It also solves $u'' = -(V_*)_u(u)$ by (33).

The leaves are written explicitly. The integers are equilibrium points, so

$$(51) \quad U_0(x, k) = k \quad (k \in \mathbb{Z}).$$

For $t \in (k, k+1)$, separation of variables gives

$$(52) \quad \cot(\pi U_0(x, t)) = \cot(\pi t) - \pi x.$$

Equivalently, every nonconstant leaf is a translate of

$$(53) \quad u_{k,s}(x) = k + \frac{1}{2} + \frac{1}{\pi} \arctan(\pi(x-s)).$$

Indeed,

$$\cot(\pi u_{k,s}(x)) = \pi(s-x).$$

The leaf $u_{k,s}$ increases from k to $k+1$:

$$(54) \quad \lim_{x \rightarrow -\infty} u_{k,s}(x) = k, \quad \lim_{x \rightarrow +\infty} u_{k,s}(x) = k+1.$$

It is bounded, and the constant leaves are bounded as well. Therefore every leaf has rotation number zero.

The map

$$(x, t) \mapsto (x, U_0(x, t))$$

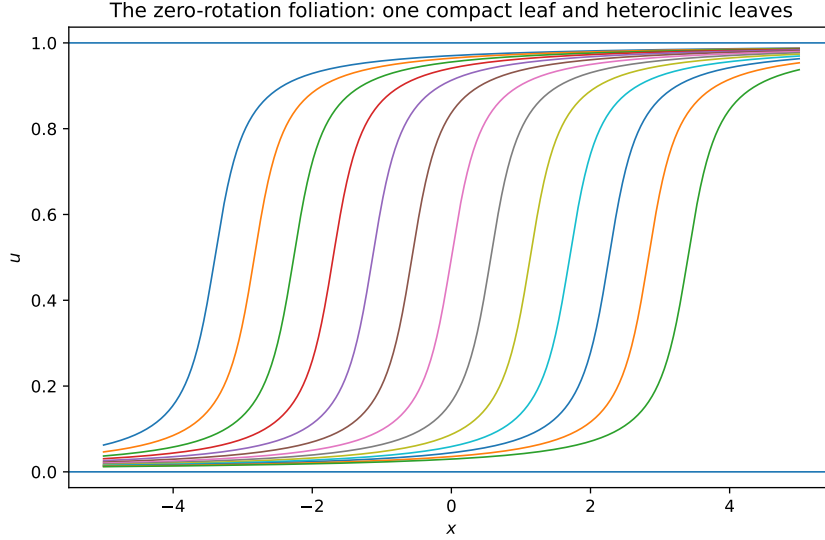


Figure 4. The zero-rotation foliation on the universal cover. The horizontal integer leaves project to the same compact leaf of the torus, while every other leaf is heteroclinic between consecutive lifts of that compact leaf. The Python plotting code was developed with assistance from Codex; see the AI use statement.

is a smooth global diffeomorphism of the lifted plane because its Jacobian is $\partial_t U_0 > 0$. Equations (46) and (47) show that the deck group maps leaves to leaves. Hence this is an ordinary minimizing foliation of $\mathbb{T}^2$. It is the Reeb-type foliation shown in Figure 4.

2.7. *The zero holonomy is nontrivial.* Using the flow law,

$$U_0(x+1, t) = \Phi_{x+1}(t) = \Phi_x(\Phi_1(t)).$$

Hence the primitive zero-rotation holonomy is

$$(55) \quad \mathfrak{h}_{0,1}(t) = \Phi_1(t).$$

It fixes every integer. On each interval $(k, k+1)$, the vector field b is strictly positive, so

$$(56) \quad \mathfrak{h}_{0,1}(t) > t \quad (t \in (k, k+1)).$$

In particular,

$$\mathfrak{h}_{0,1} \neq \text{Id}.$$

The explicit formula is obtained from (52) at $x = 1$:

$$(57) \quad \cot(\pi \mathfrak{h}_{0,1}(t)) = \cot(\pi t) - \pi \quad (t \notin \mathbb{Z}).$$

Figure 5 compares this map with the identity.

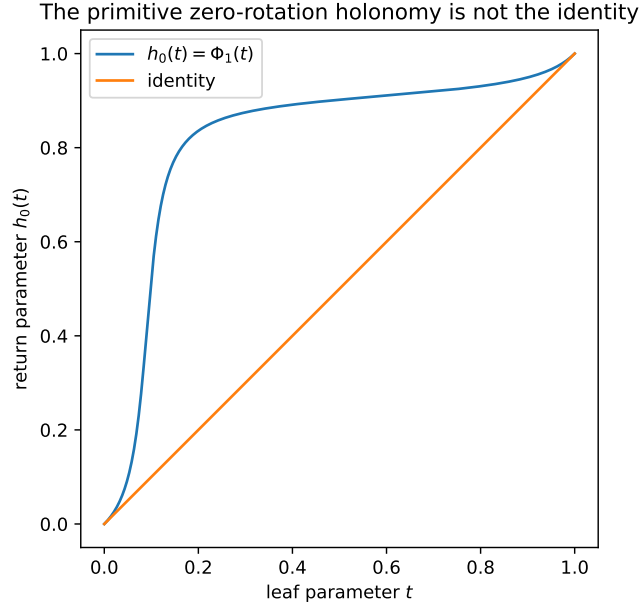


Figure 5. The primitive return map of the zero-rotation foliation. It fixes the compact leaf and moves every heteroclinic leaf forward. Since phase reparametrization only conjugates the return map, this nonidentity holonomy cannot be normalized away. The Python plotting code was developed with assistance from Codex; see the AI use statement.

We can now close the counterexample theorem.

Proof of Theorem 1.1. Equation (24) gives $(V_*)_u \not\equiv 0$. For each $\rho \neq 0$, equations (40)–(43) give an ordered deck-invariant foliation by class-A minimizers of rotation ρ , and (44) gives identity primitive holonomy at every nonzero rational rotation. Equations (49)–(54) give the zero-rotation minimizing foliation. Its primitive holonomy is $\Phi_1 \neq \text{Id}$ by (56). Finally, (21) shows that no admissible relabelling can turn it into identity holonomy. $\square$

LEMMA 2.7 (No identity-holonomy zero foliation for the counterexample). *Every 1-periodic class-A minimizer for V_* is a constant integer. Consequently this potential admits no identity-holonomy minimizing foliation of rotation zero.*

Proof. Let u be such a minimizer and put

$$e = \int_0^1 \left(\frac{1}{2} u'^2 + \frac{1}{2} \sin^4(\pi u) \right) dx \geq 0.$$

For an integer $N \geq 3$, periodicity gives $\mathcal{A}_{V_*}[u; 0, N] = Ne$. Fix an integer k . Compare with a Sobolev curve which joins $u(0)$ to k linearly on $[0, 1]$, stays at k on $[1, N - 1]$, and joins k back to $u(0) = u(N)$ linearly on $[N - 1, N]$. Its action is bounded by a constant independent of N , because the middle part has zero action. Class-A minimality gives $Ne \leq C$ for every N , hence $e = 0$. The two nonnegative terms in the integrand therefore vanish, forcing $u' = 0$ and $u \in \mathbb{Z}$.

Identity primitive holonomy at rotation zero makes every leaf 1-periodic. Constant integer leaves do not fill the configuration plane, so they cannot constitute such a foliation. $\square$

2.8. *Why the product-torus reduction fails.* The counterexample also identifies the hidden assumption in the positive argument. For this calculation, suppose the ordinary zero foliation is smooth, with $U_t > 0$, and is normalized by $U(0, t) = t$. These properties hold for the counterexample just constructed; they are not assumed for every ordinary foliation in Definition 2.2. Its holonomy $h(t) = U(1, t)$ is then smooth. The deck law is

$$(58) \quad U(x + 1, t) = U(x, h(t)).$$

Differentiating in t gives

$$(59) \quad U_t(x + 1, t) = U_t(x, h(t))h'(t).$$

Therefore the coefficient

$$a(x, t) := U_t(x, t)^2$$

satisfies

$$(60) \quad a(x + 1, t) = a(x, h(t))h'(t)^2.$$

It is the transformation law of a quadratic density on the mapping torus of h , and unless $h = \text{Id}$, this is not the periodicity law of a scalar function on the product torus $\mathbb{T}_x \times \mathbb{T}_t$.

For the counterexample, $h = \Phi_1 \neq \text{Id}$. Hence the step

$$a(x, t) = U_t(x, t)^2 \in C^\infty(\mathbb{T}^2)$$

is false under the ordinary hypothesis. This is the point repaired by the identity-holonomy assumption in the rest of the paper.

3. Identity holonomy and the zero-foliation reduction

From this point we work under the hypotheses of Theorem 1.2: every primitive rational class has an ordinary minimizing foliation and its primitive holonomy is the identity.

3.1. The rigid rational hypothesis.

Definition 3.1 (Identity-holonomy rational foliation). Fix a primitive pair $(m, \ell) \in \mathbb{Z} \times \mathbb{N}$. An *identity-holonomy lifted foliation of rotation $-m/\ell$* is an ordinary foliation in the sense of Definition 2.2 whose primitive holonomy is the identity. Equivalently (after the vertical deck action has been normalized) it is a continuous map

$$U_{m,\ell} : \mathbb{R}_x \times \mathbb{R}_t \longrightarrow \mathbb{R}$$

such that

$$(61) \quad \begin{aligned} U_{m,\ell}(x, t+1) &= U_{m,\ell}(x, t) + 1, \\ U_{m,\ell}(x + \ell, t) &= U_{m,\ell}(x, t) - m, \end{aligned}$$

for every (x, t) , the maps $t \mapsto U_{m,\ell}(x, t)$ are strictly increasing homeomorphisms, the family of graphs is invariant under the full integer deck group, and every leaf is a class-A minimizing solution of (2).

The theorem assumes one such foliation for every primitive pair. The choices for different pairs are independent. In particular, we do not assume any continuity or other regularity of the assignment $\rho \mapsto U_\rho$.

The zero-rotation class is $(m, \ell) = (0, 1)$ and therefore $\rho = 0$. We define

$$(62) \quad U_0 := U_{0,1}.$$

After normalizing the leaf parameter by its value at $x = 0$, the deck laws are

$$(63) \quad \begin{aligned} U_0(0, c) &= c, \\ U_0(x + 1, c) &= U_0(x, c), \\ U_0(x, c + 1) &= U_0(x, c) + 1. \end{aligned}$$

The second line is the assumed new condition $\mathfrak{h}_{0,1} = \text{Id}$ that we need for the positive result and not a consequence of the ordinary foliation axioms, as we developed in Section 2.

3.2. Automatic transverse regularity. Only continuity in the leaf parameter is part of Definition 3.1. The differentiability used below is a consequence.

PROPOSITION 3.2 (Automatic transverse regularity). *Let $U_{m,\ell}$ be an identity-holonomy rational minimizing foliation. After the phase normalization*

$$(64) \quad U_{m,\ell}(0, t) = t,$$

one has

$$U_{m,\ell} \in C^\infty(\mathbb{R}^2), \quad \partial_t U_{m,\ell} > 0.$$

Moreover, $U_{m,\ell,t} = \partial_t U_{m,\ell}$ is ℓ -periodic in x and 1-periodic in t .

Proof. This is proved in Appendix B. The central object is the positive finite difference

$$\frac{U(x, t + s) - U(x, t)}{s}.$$

The identity-holonomy deck law makes this function ℓ -periodic in x . It solves a scalar linear equation with uniformly bounded coefficients, so a periodic Riccati estimate gives compactness. Every limit is a positive periodic Jacobi field normalized to equal one at $x = 0$. The Wronskian makes that normalized field unique, and the difference quotients therefore converge without taking a subsequence. The positive Jacobi field makes the Dirichlet linearization invertible; Lemma B.3 then supplies the smooth bootstrap. $\square$

In particular, for the zero foliation,

$$(65) \quad U_{0,c}(x, c) > 0.$$

3.3. *The Hilbert field of an identity-holonomy foliation.* Let $U = U_{m,\ell}$ be one normalized rational foliation. For each $(x, u) \in \mathbb{R}^2$, there is a unique inverse phase $T(x, u)$ satisfying

$$(66) \quad U(x, T(x, u)) = u.$$

Define the leaf slope field

$$(67) \quad b_U(x, u) := U_x(x, T(x, u)).$$

Because of (61), this field is smooth on the rational quotient determined by

$$(x, u) \sim (x + \ell, u - m), \quad (x, u) \sim (x, u + 1).$$

Along a leaf,

$$b_U(x, U(x, t)) = U_x(x, t).$$

Differentiate this identity in x while keeping t fixed. Since $U_x = b_U(x, U)$,

$$(68) \quad \begin{aligned} U_{xx} &= (b_U)_x(x, U) + (b_U)_u(x, U)U_x \\ &= (b_U)_x(x, U) + b_U(x, U)(b_U)_u(x, U). \end{aligned}$$

Every point belongs to exactly one leaf, and every leaf solves $U_{xx} = Q(x, U)$. Hence on the whole quotient

$$(69) \quad (b_U)_x + b_U(b_U)_u = Q = -V_u.$$

Define the Hilbert one-form

$$(70) \quad \omega_U := b_U(x, u) du - \left(\frac{1}{2} b_U(x, u)^2 + V(x, u) \right) dx.$$

LEMMA 3.3 (Closed Hilbert form and exact square). *The form ω_U is closed. Moreover, for every absolutely continuous graph $u = w(x)$,*

$$(71) \quad \left(\frac{1}{2} w'^2 - V(x, w) \right) dx = \omega_U|_{\text{graph } w} + \frac{1}{2} (w' - b_U(x, w))^2 dx.$$

On a leaf $w(x) = U(x, t)$, the square vanishes and

$$(72) \quad \omega_U|_{u=U(x, t)} = \left(\frac{1}{2} U_x(x, t)^2 - V(x, U(x, t)) \right) dx.$$

Proof. From (69),

$$(73) \quad \begin{aligned} d\omega_U &= (b_{U,x} + b_U b_{U,u} + V_u) dx \wedge du \\ &= 0. \end{aligned}$$

On the graph of w ,

$$\omega_U = \left(b_U(x, w) w' - \frac{1}{2} b_U(x, w)^2 - V(x, w) \right) dx.$$

Subtracting this expression from the Lagrangian density gives $\frac{1}{2}(w' - b_U)^2 dx$, which is (71). A leaf has $U_x = b_U(x, U)$, so the square vanishes. $\square$

The weak Stokes and Sobolev details are in Appendix E. The direct consequences are worth stating here.

COROLLARY 3.4 (Calibration consequences). *Every leaf of an identity-holonomy rational foliation has the following properties.*

- (i) *Every finite segment minimizes the original action among all curves with the same endpoints.*
- (ii) *On the rational quotient, every closed leaf minimizes the action among all closed graphs in the same homology class.*
- (iii) *The action per primitive cell is independent of the leaf parameter.*

Proof. For fixed endpoints, the integral of the closed form is the same on the leaf and on any competitor, while the square in (71) is nonnegative. The same argument on the rational quotient compares homologous closed graphs. Finally, all leaves of the foliation are homologous closed graphs because $\mathfrak{h}_{m,\ell} = \text{Id}$, so their calibrated periods agree. $\square$

In particular,

COROLLARY 3.5 (Phase-independence of rational leaf action). *For an identity-holonomy rational foliation $U_{m,\ell}$, the quantity*

$$(74) \quad \frac{1}{\ell} \int_0^\ell \left(\frac{1}{2} U_{m,\ell,x}(x, t)^2 - V(x, U_{m,\ell}(x, t)) \right) dx$$

is independent of t .

3.4. *The zero inverse phase and the product torus.* We now use the zero foliation. Since $U_{0,c} > 0$, for each (x, u) there is a unique $c = T_0(x, u)$ with

$$(75) \quad U_0(x, T_0(x, u)) = u.$$

The inverse is smooth. From (63),

$$(76) \quad \begin{aligned} T_0(x+1, u) &= T_0(x, u), \\ T_0(x, u+1) &= T_0(x, u) + 1. \end{aligned}$$

The first equality uses $\mathfrak{h}_{0,1} = \text{Id}$ and is the point which fails in the counterexample.

Define

$$(77) \quad b_0(x, u) := U_{0,x}(x, T_0(x, u))$$

and

$$(78) \quad \omega_0 := b_0(x, u) du - \left(\frac{1}{2} b_0(x, u)^2 + V(x, u) \right) dx.$$

By Lemma 3.3, $d\omega_0 = 0$.

The coordinate map

$$(79) \quad \Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \Psi(x, c) = (x, U_0(x, c)),$$

has Jacobian determinant $U_{0,c} > 0$. It is proper and bijective because, for each x , $c \mapsto U_0(x, c)$ is an increasing lift onto $\mathbb{R}$. Hence it is a global diffeomorphism. The deck laws show that it is $\mathbb{Z}^2$ -equivariant and descends to a diffeomorphism of the product torus.

We now use the zero foliation as a coordinate system. Since $U_{0,c} > 0$, for each (x, u) there is a unique $c = T_0(x, u)$ such that $U_0(x, T_0(x, u)) = u$. Thus $\Psi(x, c) := (x, U_0(x, c))$ defines a smooth change of coordinates. In these coordinates,

$$du = U_{0,x}(x, c) dx + U_{0,c}(x, c) dc.$$

Since

$$b_0(x, U_0(x, c)) = U_{0,x}(x, c),$$

the pullback of the Hilbert form

$$\omega_0 = b_0(x, u) du - \left(\frac{1}{2} b_0(x, u)^2 + V(x, u) \right) dx$$

is

$$\Psi^* \omega_0 = \left(\frac{1}{2} U_{0,x}(x, c)^2 - V(x, U_0(x, c)) \right) dx + U_{0,x}(x, c) U_{0,c}(x, c) dc.$$

3.5. *The exact zero-foliation square identity.* Let $c \in W_{\text{loc}}^{1,2}(\mathbb{R})$ and define

$$(80) \quad u(x) := U_0(x, c(x)).$$

The Sobolev chain rule gives, almost everywhere,

$$(81) \quad \begin{aligned} u' &= U_{0,x}(x, c) + U_{0,c}(x, c)c' \\ &= b_0(x, u) + U_{0,c}(x, c)c'. \end{aligned}$$

PROPOSITION 3.6 (Zero-foliation square identity). *For u and c related by (80),*

$$(82) \quad \begin{aligned} \left(\frac{1}{2}u'^2 - V(x, u) \right) dx &= \omega_0|_{\text{graph } u} \\ &\quad + \frac{1}{2}a(x, c)c'^2 dx, \end{aligned}$$

where

$$(83) \quad a(x, c) := U_{0,c}(x, c)^2.$$

The coefficient a is smooth, positive, and 1-periodic in both variables.

Proof. Apply (71) to the zero field. By (81),

$$u' - b_0(x, u) = U_{0,c}(x, c)c'.$$

This proves the square identity. Positivity follows from (65). The vertical deck law gives $U_{0,c}(x, c+1) = U_{0,c}(x, c)$, and identity zero holonomy gives $U_{0,c}(x+1, c) = U_{0,c}(x, c)$. Hence a is a coefficient on the product torus. $\square$

Without identity zero holonomy, Proposition 3.6 still holds on the universal cover, but a obeys the mapping-torus law (60). The finite-amplitude theorem below is a product-torus theorem.

We therefore define the reduced Lagrangian

$$L_a(x, c, v) := \frac{1}{2}a(x, c)v^2, \quad a(x, c) := U_{0,c}(x, c)^2,$$

and the corresponding action on an interval $I = [r, s]$ by

$$\mathcal{A}_a[c; I] := \frac{1}{2} \int_r^s a(x, c(x)) c'(x)^2 dx.$$

3.6. *Fixed-endpoint variational equivalence.*

PROPOSITION 3.7 (Variational equivalence). *Let $I = [r, s]$. Suppose $c, \tilde{c} \in W^{1,2}(I)$ have the same endpoint values, and set*

$$u = U_0(\cdot, c), \quad \tilde{u} = U_0(\cdot, \tilde{c}).$$

Then

$$(84) \quad \begin{aligned} \mathcal{A}_V[\tilde{u}; r, s] - \mathcal{A}_V[u; r, s] &= \frac{1}{2} \int_r^s a(x, \tilde{c}) \tilde{c}'^2 dx \\ &\quad - \frac{1}{2} \int_r^s a(x, c) c'^2 dx. \end{aligned}$$

Consequently u is fixed-endpoint minimizing on I if and only if c is fixed-endpoint minimizing for $L_a = \frac{1}{2}av^2$ on I . The same equivalence holds for class-A minimality.

Proof. The two physical graphs have the same endpoints. Since ω_0 is closed, their ω_0 integrals agree. Subtract the two instances of (82). Appendix E gives the weak Stokes approximation for Sobolev graphs, and Appendix A gives the compact-support equivalence. $\square$

Hence no minimizing information is lost in the coordinate change.

3.7. *Transfer of a rigid rational foliation.* Fix a primitive pair (m, ℓ) and its identity-holonomy foliation $U_{m, \ell}$. Define its representation in the zero coordinate by

$$(85) \quad C_{m, \ell}(x, t) := T_0(x, U_{m, \ell}(x, t)).$$

The two relevant deck calculations are

$$(86) \quad \begin{aligned} C_{m, \ell}(x, t+1) &= C_{m, \ell}(x, t) + 1, \\ C_{m, \ell}(x + \ell, t) &= C_{m, \ell}(x, t) - m. \end{aligned}$$

Indeed, the first line follows from the vertical decks of $U_{m, \ell}$ and T_0 . For the second,

$$\begin{aligned} C_{m, \ell}(x + \ell, t) &= T_0(x + \ell, U_{m, \ell}(x + \ell, t)) \\ &= T_0(x, U_{m, \ell}(x, t) - m) \\ &= C_{m, \ell}(x, t) - m. \end{aligned}$$

Here we used both $T_0(x + 1, u) = T_0(x, u)$ and $U_{m, \ell}(x + \ell, t) = U_{m, \ell}(x, t) - m$.

This is the second exact use of identity holonomy. Under ordinary holonomy, the correct formula would be

$$(87) \quad C_{m, \ell}(x + \ell, t) = C_{m, \ell}(x, \mathfrak{h}_{m, \ell}(t)) - m,$$

so a fixed leaf would not in general belong to a closed winding class.

By Proposition 3.7, every transformed leaf is class-A minimizing for L_a . More strongly, it minimizes its full rational winding class.

PROPOSITION 3.8 (Full winding minimality of transformed rational leaves). *Fix t and let*

$$c_t = C_{m, \ell}(\cdot, t), \quad u_t = U_{m, \ell}(\cdot, t).$$

Then, for every c satisfying $c(x + \ell) = c(x) - m$,

$$(88) \quad \frac{1}{2\ell} \int_0^\ell a(x, c_t) c_t'^2 dx \leq \frac{1}{2\ell} \int_0^\ell a(x, c) c'^2 dx.$$

Proof. Let $u(x) = U_0(x, c(x))$. By the deck laws of U_0 , the graph of u has the same rational homology class $(\ell, -m)$ as the graph of u_t . The Hilbert form $\omega_{U_{m,\ell}}$ calibrates u_t . Therefore

$$(89) \quad \mathcal{A}_V[u; 0, \ell] \geq \mathcal{A}_V[u_t; 0, \ell]$$

by Corollary 3.4(ii).

The zero form ω_0 is also closed, and the two graphs are homologous. Hence its periods agree. Subtracting the zero-foliation square identities for u and u_t gives

$$(90) \quad \begin{aligned} \frac{1}{2} \int_0^\ell a(x, c) c'^2 dx - \frac{1}{2} \int_0^\ell a(x, c_t) c_t'^2 dx \\ = \mathcal{A}_V[u; 0, \ell] - \mathcal{A}_V[u_t; 0, \ell] \geq 0. \end{aligned}$$

This is (88) as required. $\square$

The preceding proposition is the required cell-minimization statement. It is stronger than fixed-endpoint minimality and is why we kept both calibrating forms in the calculation.

3.8. *The phase map and flat reduced cells.* For a primitive pair define

$$(91) \quad \Pi_{m,\ell}(c) := \frac{1}{\ell} \int_0^\ell (mx + \ell c(x)) dx \pmod{\ell}.$$

This agrees with (9). Let

$$(92) \quad \Theta_{m,\ell}(t) := \Pi_{m,\ell}(C_{m,\ell}(\cdot, t)) \in \mathbb{T}_\ell.$$

PROPOSITION 3.9 (The transformed phase map). *The map $\Theta_{m,\ell}$ is an orientation-preserving circle homeomorphism of degree one. In particular, it is onto $\mathbb{T}_\ell$.*

Proof. The transformed leaves are strictly ordered because both $U_{m,\ell}$ and $T_0(x, \cdot)$ are strictly increasing. Hence $t_1 < t_2$ implies

$$C_{m,\ell}(x, t_1) < C_{m,\ell}(x, t_2) \quad \text{for every } x.$$

Taking phase means preserves the strict order. The vertical deck law gives

$$\Theta_{m,\ell}(t + 1) = \Theta_{m,\ell}(t) + \ell.$$

Therefore the induced map $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\ell\mathbb{Z}$ is strictly increasing and has degree one. It is consequently a circle homeomorphism. $\square$

PROPOSITION 3.10 (Flatness inherited from identity-holonomy rational foliations). *The reduced coefficient $a = U_{0,c}^2$ has flat gauge-fixed rational cells.*

Proof. Fix a primitive pair (m, ℓ) . By Proposition 3.8, every transformed leaf c_t minimizes the reduced action over the entire winding class $c(x + \ell) = c(x) - m$. Hence all of these leaves have the same value, namely the minimum over that full class. By Proposition 3.9, every phase $Y \in \mathbb{T}_\ell$ is represented by exactly one transformed leaf. Therefore $E_{m,\ell}(Y)$ equals the same full-class minimum for every Y . The cell is flat. Since the primitive pair was arbitrary, every rational cell is flat. $\square$

We have now reduced Theorem 1.2 to the independent finite-amplitude classification of Theorem 1.4. The remaining sections no longer refer to the original holonomy maps.

4. The gauge-fixed resonant cell problem

From this section through Section 12, a denotes an arbitrary fixed element of $C^\infty(\mathbb{T}^2)$ with

$$(93) \quad 0 < a_- \leq a(x, c) \leq a_+ < \infty.$$

We study the Lagrangian $L_a = \frac{1}{2}av^2$ independently of its origin from a potential.

4.1. *The rational winding space.* For a primitive pair (m, ℓ) , define

$$(94) \quad \mathcal{C}_{m,\ell} := \left\{ c \in W_{\text{loc}}^{1,2}(\mathbb{R}) : c(x + \ell) = c(x) - m \right\}.$$

Every element is determined by its restriction to $[0, \ell]$, whose endpoint traces satisfy $c(\ell) = c(0) - m$. The normalized cell action is

$$(95) \quad \mathcal{A}_{m,\ell}(c) := \frac{1}{2\ell} \int_0^\ell a(x, c(x)) c'(x)^2 dx.$$

The resonant phase $Z = mx + \ell c$ is ℓ -periodic, and

$$(96) \quad \Pi_{m,\ell}(c) := \frac{1}{\ell} \int_0^\ell Z(x) dx \pmod{\ell}.$$

Vertical translation $c \mapsto c + 1$ preserves the action and changes a lift of the phase by ℓ , so the phase belongs to $\mathbb{T}_\ell$.

4.2. *Why the ungauged problem is degenerate.* Write

$$(97) \quad c(x) = \frac{Y + \varphi(x) - mx}{\ell}, \quad \varphi(x + \ell) = \varphi(x).$$

Then

$$c'(x) = \frac{\varphi'(x) - m}{\ell}$$

and

$$(98) \quad \mathcal{A}_{m,\ell}(c) = \frac{1}{2\ell^3} \int_0^\ell a\left(x, \frac{Y + \varphi(x) - mx}{\ell}\right) (\varphi'(x) - m)^2 dx.$$

If φ is allowed an arbitrary constant mode, Y is not fixed.

PROPOSITION 4.1 (Degeneracy of the ungauged phase). *Define*

$$\widetilde{E}_{m,\ell}(Y) := \inf_{\varphi \in W_{\text{per}}^{1,2}(0,\ell)} \frac{1}{2\ell^3} \int_0^\ell a\left(x, \frac{Y + \varphi - mx}{\ell}\right) (\varphi' - m)^2 dx.$$

Then $\widetilde{E}_{m,\ell}$ is constant in Y for every positive periodic coefficient a .

Proof. Given Y_0, Y_1 and a competitor φ at Y_0 , set $\widetilde{\varphi} = \varphi + Y_0 - Y_1$. Its derivative is unchanged and

$$Y_1 + \widetilde{\varphi} - mx = Y_0 + \varphi - mx.$$

Hence the two integrands agree pointwise. Taking infima in both directions proves equality. $\square$

This proposition explains why the phase constraint in (10) cannot be replaced by an informal “phase label.” A gauge condition must remove the constant corrector.

4.3. *The mean-zero corrector gauge.* Let

$$(99) \quad W_{\text{per},0}^{1,2}(0,\ell) := \left\{ \varphi \in W_{\text{per}}^{1,2}(0,\ell) : \frac{1}{\ell} \int_0^\ell \varphi dx = 0 \right\}.$$

LEMMA 4.2 (Unique resonant decomposition). *Every $c \in \mathcal{C}_{m,\ell}$ has a unique representation*

$$(100) \quad mx + \ell c(x) = Y + \varphi(x),$$

where $Y \in \mathbb{R}$ and $\varphi \in W_{\text{per},0}^{1,2}(0,\ell)$. One has

$$Y = \frac{1}{\ell} \int_0^\ell (mx + \ell c(x)) dx.$$

Conversely, every such pair (Y, φ) defines an element of $\mathcal{C}_{m,\ell}$ by (97).

Proof. The resonant phase $Z = mx + \ell c$ is periodic. Decompose it into its mean and mean-zero part. Uniqueness follows by taking means. Conversely, periodicity of φ gives

$$c(x + \ell) = \frac{Y + \varphi(x) - m(x + \ell)}{\ell} = c(x) - m.$$

$\square$

Consequently the cell energy can be written as

$$(101) \quad E_{m,\ell}(Y) = \inf_{\varphi \in W_{\text{per},0}^{1,2}(0,\ell)} \frac{1}{2\ell^3} \int_0^\ell a\left(x, \frac{\widetilde{Y} + \varphi(x) - mx}{\ell}\right) (\varphi'(x) - m)^2 dx,$$

where $\widetilde{Y}$ is any lift of $Y \in \mathbb{T}_\ell$. Replacing $\widetilde{Y}$ by $\widetilde{Y} + \ell$ translates c by one and leaves the integrand unchanged.

4.4. Existence and regularity.

PROPOSITION 4.3 (Well-posedness of the gauge-fixed cell). *For every primitive (m, ℓ) and every $Y \in \mathbb{T}_\ell$, the infimum in (10) is attained. Every minimizer is smooth and satisfies the constrained Euler–Lagrange equation*

$$(102) \quad -\frac{d}{dx}(a(x, c)c'(x)) + \frac{1}{2}a_c(x, c)c'(x)^2 = \lambda$$

for a constant multiplier $\lambda \in \mathbb{R}$. The equation holds on $\mathbb{R}$ after periodic extension, and $c(x + \ell) = c(x) - m$.

Proof. The direct method and the derivation of the multiplier are given in full in Appendix C. Briefly, the lower bound on a controls φ' in (101); Poincaré's inequality controls the mean-zero corrector in $W^{1,2}$; compact embedding gives uniform convergence of a minimizing sequence; and weak lower semicontinuity gives a minimizer. The first variation vanishes against periodic zero-mean variations. The annihilator of that subspace consists of constants, which produces λ . One-dimensional regularity bootstraps the weak equation to C^∞ . $\square$

Remark 4.4 (Meaning of the multiplier). The multiplier measures the response of the minimized cell action to the phase constraint. If a smooth local branch $Y \mapsto c_Y$ is chosen with a compatible lift, the envelope identity is

$$\frac{d}{dY}E_{m,\ell}(Y) = \frac{\lambda_Y}{\ell}$$

under the convention in (102). We will not need differentiability of the branch: under flatness, a simpler comparison proves directly that $\lambda = 0$.

4.5. Basic bounds and continuity. The cell energies are finite because the affine curve $c(x) = c_0 - mx/\ell$ is admissible after choosing c_0 to match the phase. The lower bound in (93) gives coercivity.

PROPOSITION 4.5 (Continuity of the cell energy). *The function $Y \mapsto E_{m,\ell}(Y)$ is continuous on $\mathbb{T}_\ell$. On compact sets of positive coefficients it is locally Lipschitz with a constant depending only on finitely many C^1 bounds for a , on a_-^{-1} , and on (m, ℓ) .*

Proof. Let c_Y be a minimizer and compare at $Y + h$ with $c_Y + h/\ell$. This translation changes the phase by h , preserves the derivative, and changes the coefficient by at most $C|h|$. The energy of c_Y is uniformly bounded by an affine competitor. This gives the one-sided Lipschitz bound; interchanging Y and $Y + h$ gives the other side. Details, including the uniform gradient bound needed in the estimate, appear in Appendix C. $\square$

4.6. *Flatness and its immediate consequence.* Assume now that the (m, ℓ) cell is flat:

$$(103) \quad E_{m,\ell}(Y) = e_{m,\ell} \quad \text{for every } Y \in \mathbb{T}_\ell.$$

PROPOSITION 4.6 (Flatness removes the phase constraint). *Let c_Y minimize the gauge-fixed problem at phase Y . Then*

$$(104) \quad \mathcal{A}_{m,\ell}(c_Y) = \inf_{c \in \mathcal{C}_{m,\ell}} \mathcal{A}_{m,\ell}(c).$$

In particular the multiplier in (102) vanishes and c_Y solves the unforced equation

$$(105) \quad \frac{d}{dx} (a(x, c)c') - \frac{1}{2}a_c(x, c)c'^2 = 0.$$

Proof. Every $c \in \mathcal{C}_{m,\ell}$ has a phase $\Pi_{m,\ell}(c)$. Hence

$$\mathcal{A}_{m,\ell}(c) \geq E_{m,\ell}(\Pi_{m,\ell}(c)) = e_{m,\ell} = \mathcal{A}_{m,\ell}(c_Y).$$

Hence c_Y minimizes over the full winding class. Arbitrary periodic variations are now admissible, so the unconstrained Euler–Lagrange equation holds. $\square$

This elementary observation is the point at which phase flatness becomes a dynamical statement.

5. Separable coefficients have flat rational cells

We prove the easy direction of Theorem 1.4 and record the exact minimizing leaves. Let

$$(106) \quad a(x, c) = A(x)B(c), \quad A > 0, \quad B > 0,$$

with A and B smooth and 1-periodic.

5.1. *The vertical Liouville coordinate.* Define

$$(107) \quad h(c) := \int_0^c \sqrt{B(s)} ds, \quad \beta := \int_0^1 \sqrt{B(s)} ds.$$

Then $h' > 0$ and

$$(108) \quad h(c+1) = h(c) + \beta.$$

Hence h is a lift of an orientation-preserving diffeomorphism between circles of lengths 1 and β .

For a curve $c(x)$ let $z(x) = h(c(x))$. Then

$$(109) \quad B(c)c'^2 = z'^2,$$

and the winding law $c(x + \ell) = c(x) - m$ becomes

$$(110) \quad z(x + \ell) = z(x) - m\beta.$$

The action is

$$(111) \quad \mathcal{A}_{m,\ell}(c) = \frac{1}{2\ell} \int_0^\ell A(x)z'(x)^2 dx.$$

5.2. *Explicit weighted Dirichlet minimizers.* Let

$$(112) \quad J_A(\ell) := \int_0^\ell A(t)^{-1} dt.$$

For $s \in \mathbb{R}$, define

$$(113) \quad z_s(x) := s - m\beta \frac{\int_0^x A(t)^{-1} dt}{J_A(\ell)}.$$

Then

$$A(x)z'_s(x) = -\frac{m\beta}{J_A(\ell)}$$

is constant, so z_s is the unique minimizer of (111) in its winding class with prescribed additive constant. Its action is

$$(114) \quad e_{m,\ell}^{\text{sep}} = \frac{m^2\beta^2}{2\ell J_A(\ell)}.$$

LEMMA 5.1 (Weighted Dirichlet inequality). *Every absolutely continuous z satisfying (110) obeys*

$$(115) \quad \frac{1}{2\ell} \int_0^\ell Az'^2 dx \geq \frac{m^2\beta^2}{2\ell J_A(\ell)},$$

with equality if and only if Az' is constant almost everywhere.

Proof. Cauchy–Schwarz gives

$$\begin{aligned} m^2\beta^2 &= \left(\int_0^\ell z' dx \right)^2 \\ &\leq \left(\int_0^\ell Az'^2 dx \right) \left(\int_0^\ell A^{-1} dx \right). \end{aligned}$$

Equality holds exactly when $\sqrt{A}z'$ is proportional to $A^{-1/2}$. □

Set

$$(116) \quad c_s(x) := h^{-1}(z_s(x)).$$

These are smooth ordered minimizers in $\mathcal{C}_{m,\ell}$ and all have action (114).

5.3. *The phase map.* Define

$$(117) \quad \Theta(s) := \Pi_{m,\ell}(c_s).$$

Since h^{-1} is strictly increasing, $s_1 < s_2$ implies $c_{s_1}(x) < c_{s_2}(x)$ for every x , hence Θ is strictly increasing. Moreover

$$z_{s+\beta} = z_s + \beta, \quad c_{s+\beta} = c_s + 1,$$

so

$$(118) \quad \Theta(s + \beta) = \Theta(s) + \ell.$$

Therefore Θ induces an orientation-preserving circle homeomorphism from $\mathbb{R}/\beta\mathbb{Z}$ to $\mathbb{T}_\ell$.

PROPOSITION 5.2 (Separable flatness). *If $a(x, c) = A(x)B(c)$, then every primitive rational cell is flat and*

$$(119) \quad E_{m,\ell}(Y) = \frac{m^2 \beta^2}{2\ell \int_0^\ell A(t)^{-1} dt} \quad \text{for all } Y \in \mathbb{T}_\ell.$$

Proof. By the surjectivity of Θ , every phase Y is represented by a curve c_s . Lemma 5.1 shows that c_s minimizes over the full winding class, hence also over the prescribed phase class. Its action is independent of s and equals (114). $\square$

Remark 5.3 (The zero-winding cell). When $m = 0$ and primitivity forces $\ell = 1$, the minimum is zero and is attained by all constant curves. The phase map is simply the vertical Liouville diffeomorphism. The formula above remains valid.

6. Flat rational cells generate minimizing foliations

Fix a primitive pair (m, ℓ) and assume the cell is flat. We prove that the gauge-phase minimizers form a complete rational foliation of the base, then lift it to an invariant graph for the Euler–Lagrange flow.

6.1. Crossing and uniqueness. Let $\mathcal{M}_{m,\ell}$ denote the set of minimizers of $\mathcal{A}_{m,\ell}$ over the full winding class $\mathcal{C}_{m,\ell}$. Proposition 4.6 shows that every constrained phase minimizer belongs to $\mathcal{M}_{m,\ell}$.

LEMMA 6.1 (No crossing in one rational class). *If $c_1, c_2 \in \mathcal{M}_{m,\ell}$, then either $c_1 \equiv c_2$ or their lifted graphs are disjoint. In the latter case one of the strict inequalities $c_1 < c_2$ or $c_2 < c_1$ holds on all of $\mathbb{R}$.*

Proof. The detailed Sobolev splicing proof is in Appendix D. Define $c^- := \min(c_1, c_2)$ and $c^+ := \max(c_1, c_2)$. Both belong to the same winding class and the sum of their actions equals the sum of the actions of c_1, c_2 . Since the latter are minimal, both $c^\pm$ are minimizers. They are therefore smooth solutions of

(105). At a crossing, smoothness of the spliced minimizers forces equality of the first derivatives. ODE uniqueness then identifies the two original curves. $\square$

PROPOSITION 6.2 (Unique minimizer at each gauge phase). *For every $Y \in \mathbb{T}_\ell$ there is a unique minimizer c_Y with $\Pi_{m,\ell}(c_Y) = Y$, modulo the simultaneous convention $c_{Y+\ell} = c_Y + 1$ for lifted phases.*

Proof. Existence is Proposition 4.3. If two minimizers have the same phase, choose vertical translates so their lifted phase means agree in $\mathbb{R}$. By Lemma 6.1, distinct minimizers are strictly ordered. If $c_1 < c_2$, then

$$\Pi_{m,\ell}(c_2) - \Pi_{m,\ell}(c_1) = \int_0^\ell (c_2 - c_1) dx > 0,$$

a contradiction. $\square$

6.2. Ordering and continuity in phase. Choose the unique lifted family satisfying

$$(120) \quad c_{Y+\ell}(x) = c_Y(x) + 1.$$

PROPOSITION 6.3 (Strict phase order). *If $Y_1 < Y_2 < Y_1 + \ell$, then*

$$c_{Y_1}(x) < c_{Y_2}(x) \quad \text{for every } x.$$

Proof. The curves cannot meet by Lemma 6.1. If their order were reversed, integration would give the reverse order of their phase means. Hence the displayed order holds. $\square$

PROPOSITION 6.4 (Continuity of the phase family). *The map*

$$(x, Y) \longmapsto c_Y(x)$$

is continuous on $\mathbb{R}^2$. On compact x -intervals the convergence is C^1 when $Y_j \rightarrow Y$.

Proof. Uniform action bounds and the Tonelli a priori estimate give local C^1 precompactness of minimizers; see Appendix D. Every subsequential limit has the limiting winding and phase, is still minimizing, and therefore equals c_Y by Proposition 6.2. Hence the whole sequence converges. Joint continuity follows from the ODE and the periodicity. $\square$

PROPOSITION 6.5 (Smooth transverse parameter). *Relabel the leaves by their value at $x = 0$: for every $t \in \mathbb{R}$, let $U_{m,\ell}(\cdot, t)$ be the unique leaf satisfying $U_{m,\ell}(0, t) = t$. Then*

$$U_{m,\ell} \in C^\infty(\mathbb{R}^2), \quad \partial_t U_{m,\ell} > 0,$$

and

$$(121) \quad \begin{aligned} U_{m,\ell}(x + \ell, t) &= U_{m,\ell}(x, t) - m, \\ U_{m,\ell}(x, t + 1) &= U_{m,\ell}(x, t) + 1. \end{aligned}$$

Proof. The relabelling is legitimate because $Y \mapsto c_Y(0)$ is a strictly increasing degree-one lift. Positive finite differences of two leaves solve the linearized Euler equation in mean-value form, are ℓ -periodic in x , and equal one at $x = 0$. Periodic Harnack and ODE estimates give compactness. Every limit is a positive periodic Jacobi field. The principal periodic Jacobi field is simple, so the normalization at $x = 0$ makes the limit unique. This gives $U_t > 0$ and continuity of the first transverse derivative. The Dirichlet implicit-function argument in Lemma B.3 gives the full smooth bootstrap. Appendix E contains the complete argument. $\square$

PROPOSITION 6.6 (Full deck covariance of the flat-cell family). *For lifted gauge phases, the uniquely chosen minimizers satisfy*

$$(122) \quad c_{Y-mj+\ell k}(x) = c_Y(x + j) + k \quad (j, k \in \mathbb{Z}).$$

Consequently the union of the leaves is invariant under the full deck group of $\mathbb{R}^2 \rightarrow \mathbb{T}^2$. Its velocity and momentum fields descend to the standard configuration torus, not merely to the primitive rational cover.

Proof. The translated curve

$$\tilde{c}(x) := c_Y(x + j) + k$$

has the same winding and the same action as c_Y , because a is 1-periodic in both variables. It is therefore a full-class minimizer. Its lifted phase is computed without suppressing the constant term:

$$\begin{aligned} \frac{1}{\ell} \int_0^\ell (mx + \ell \tilde{c}(x)) dx &= \frac{1}{\ell} \int_0^\ell (m(x + j) + \ell c_Y(x + j) - mj + \ell k) dx \\ &= Y - mj + \ell k. \end{aligned}$$

Here $x \mapsto mx + \ell c_Y(x)$ is ℓ -periodic, so its integral is unchanged by the shift $x \mapsto x + j$. Uniqueness at a fixed lifted phase then gives (122).

Deck translation preserves the derivative of a curve. Since exactly one leaf passes through each base point, the derivative attached to a point is therefore unchanged by $(x, c) \mapsto (x + j, c + k)$. The same is true of the momentum because a is periodic. $\square$

6.3. *Filling the base.* For fixed x , the map $Y \mapsto c_Y(x)$ is continuous, strictly increasing, and satisfies $c_{Y+\ell}(x) = c_Y(x) + 1$. Hence it is a homeomorphism of the corresponding lifted circles.

THEOREM 6.7 (Rational base foliation from one flat cell). *If the (m, ℓ) cell is flat, the curves*

$$\{c_Y : Y \in \mathbb{R}\}$$

form a continuous foliation of the lifted configuration plane. Every leaf is smooth, solves (105), has rotation number $-m/\ell$, and minimizes the action over its complete winding class.

Proof. The only point not already proved is filling. For every fixed x , the map $Y \mapsto c_Y(x)$ is a strictly increasing proper lift because of (120). It is therefore onto $\mathbb{R}$. Uniqueness gives one leaf through every (x, c) . $\square$

6.4. *The invariant velocity and momentum graphs.* Use the smooth normalization of Proposition 6.5, and let $T_{m,\ell}(x, c)$ be its inverse leaf phase:

$$U_{m,\ell}(x, T_{m,\ell}(x, c)) = c.$$

Define

$$(123) \quad V_{m,\ell}(x, c) := \partial_x U_{m,\ell}(x, T_{m,\ell}(x, c)),$$

and the momentum

$$(124) \quad P_{m,\ell}(x, c) := a(x, c)V_{m,\ell}(x, c).$$

By Proposition 6.6, both fields satisfy

$$(125) \quad \begin{aligned} V_{m,\ell}(x + j, c + k) &= V_{m,\ell}(x, c), \\ P_{m,\ell}(x + j, c + k) &= P_{m,\ell}(x, c) \quad (j, k \in \mathbb{Z}). \end{aligned}$$

They are therefore smooth fields on the standard torus $\mathbb{T}_x \times \mathbb{T}_c$. This is stronger than the Lipschitz graph property usually supplied by Mather's graph theorem, and it comes here from the fact that the entire base is already foliated by periodic minimizers.

The Hamiltonian dual to L_a is

$$(126) \quad H(x, c, p) = \frac{p^2}{2a(x, c)}.$$

Its equations are

$$(127) \quad c_x = H_p = \frac{p}{a(x, c)}, \quad p_x = -H_c = \frac{p^2 a_c(x, c)}{2a(x, c)^2}.$$

The union of the lifted leaf trajectories is the invariant full graph

$$(128) \quad \Gamma_{m,\ell} := \{(x, c, p) : p = P_{m,\ell}(x, c)\}.$$

Every orbit on it is a periodic rational minimizer.

PROPOSITION 6.8 (Weak invariance equation). *The smooth function $P = P_{m,\ell}$ satisfies pointwise (and hence in distributions),*

$$(129) \quad P_x + H_p(x, c, P)P_c + H_c(x, c, P) = 0.$$

Proof. Tangency of the Hamiltonian vector field to $p = P(x, c)$ gives

$$p_x = P_x + P_c c_x = P_x + P_c H_p.$$

Hamilton's equation gives $p_x = -H_c$, proving the equality pointwise. The distributional statement follows immediately. $\square$

The next section extracts the global minimizing content of this invariant graph.

7. Hamiltonian calibration and rational uniqueness

A cell minimizer is initially known to minimize only among curves satisfying a rational winding constraint. We now prove a stronger statement: every finite segment of every rational leaf minimizes among all curves with the same endpoints. The mechanism is a closed Hamiltonian one-form on the invariant momentum graph.

7.1. *The graph calibration.* Fix a flat primitive cell and abbreviate

$$P = P_{m,\ell}, \quad V = V_{m,\ell}, \quad H(x, c, p) = \frac{p^2}{2a(x, c)}.$$

Define on the lifted (x, c) plane

$$(130) \quad \lambda_{m,\ell} := P(x, c) dc - H(x, c, P(x, c)) dx.$$

PROPOSITION 7.1 (Closedness of the Hamiltonian calibration). *The smooth one-form $\lambda_{m,\ell}$ is closed. Hence there exists a smooth function $S_{m,\ell} : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that*

$$(131) \quad dS_{m,\ell} = \lambda_{m,\ell}.$$

Proof. Since P is smooth,

$$\begin{aligned} d\lambda_{m,\ell} &= \left(P_x + \frac{d}{dc} H(x, c, P(x, c)) \right) dx \wedge dc \\ &= (P_x + H_p P_c + H_c) dx \wedge dc. \end{aligned}$$

The coefficient vanishes by (129). The lifted (x, c) -plane is simply connected, so the classical Poincaré lemma gives the primitive. Appendix E records the corresponding weak statement for Lipschitz graph limits. $\square$

The primitive is generally not periodic. Its deck periods record the cohomology of the invariant graph. This causes no difficulty on the universal cover, where fixed endpoints determine the calibration difference.

7.2. *Fenchel inequality.* For every (x, c, v, p) ,

$$(132) \quad L_a(x, c, v) + H(x, c, p) \geq pv,$$

with equality exactly when

$$(133) \quad p = L_v(x, c, v) = a(x, c)v.$$

For the quadratic Lagrangian this is the identity

$$(134) \quad L_a + H - pv = \frac{1}{2a}(av - p)^2.$$

THEOREM 7.2 (Calibration inequality). *Let $\gamma \in W^{1,2}(r, s)$. Then*

$$(135) \quad \int_r^s L_a(x, \gamma, \gamma') dx \geq S_{m,\ell}(s, \gamma(s)) - S_{m,\ell}(r, \gamma(r)).$$

Equality holds if and only if

$$(136) \quad \gamma'(x) = V_{m,\ell}(x, \gamma(x))$$

almost everywhere.

Proof. Apply (132) with $p = P(x, \gamma(x))$:

$$L_a(x, \gamma, \gamma') \geq P(x, \gamma)\gamma' - H(x, \gamma, P(x, \gamma)).$$

The right side is the pullback of $\lambda_{m,\ell} = dS_{m,\ell}$ along $x \mapsto (x, \gamma(x))$. Integration gives (135). Equality in Fenchel's inequality is $P = a\gamma'$, equivalent to (136). $\square$

COROLLARY 7.3 (Class-A minimality of rational leaves). *Every finite segment of every orbit on $\Gamma_{m,\ell}$ is globally minimizing among all $W^{1,2}$ curves with the same endpoints.*

Proof. A graph orbit satisfies the equality condition in Theorem 7.2. Every competitor has the same calibration difference because its endpoints agree, and therefore has no smaller action. $\square$

This conclusion is stronger than minimization over a rational cell. It is the form needed for compact limits, conjugate points, and Hopf rigidity.

7.3. *Rational full-graph uniqueness.* We need a no-gap statement at rational rotation numbers. It is useful to prove it directly from the calibrated foliation.

PROPOSITION 7.4 (Uniqueness of a full minimizing graph at the calibrated rational rotation). *Let Γ be a continuous invariant full graph for the Hamiltonian flow of L_a . Suppose every orbit in Γ is globally minimizing and has rotation number $-m/\ell$. Then*

$$\Gamma = \Gamma_{m,\ell}.$$

Proof. The full proof is in Appendix E; we give the essential argument. Use the smooth rational foliation of Proposition 6.5 as a coordinate system. Let $U(x, t)$ be normalized by $U(0, t) = t$, and let $T(x, c)$ be the inverse phase. Along an arbitrary curve $c(x)$ set $t(x) = T(x, c(x))$. Since $T_x + VT_c = 0$ and $T_c > 0$,

$$c' - V(x, c) = \frac{t'}{T_c(x, c)}.$$

The quadratic Fenchel gap relative to the calibrated graph is therefore

$$(137) \quad L_a(x, c, c') - \lambda_{m, \ell}|_{(x, c(x))} = \frac{1}{2} \frac{a(x, c)}{T_c(x, c)^2} t'(x)^2.$$

The coefficient is positive and periodic.

An orbit in the competing graph has the same rational rotation. Its return map after ℓ base periods, expressed in the t coordinate and corrected by the integer m , has translation number zero. Hence the sequence $t(n\ell)$ is bounded after an integer normalization. The calibration identity transfers global minimality to the nonnegative functional on the right of (137). On $[-N\ell, N\ell]$ compare t with the affine interpolant of its endpoint values. The competitor energy is $O(N^{-1})$ because the endpoints remain bounded. The nonnegative energy on every fixed interval must therefore vanish. Hence $t' \equiv 0$, so the orbit is one of the calibrated leaves. Since this holds through every base point of the competing full graph, the graphs coincide. $\square$

7.4. A periodic rational barrier. The conclusion used below is class-A minimality, namely fixed-endpoint minimality on every compact interval. We do not use “static” as a synonym: cohomological staticity requires a critical normalization and an additional reverse-action equality; see, for example, the distinction in [8].

There is nevertheless a direct periodic barrier statement. For a flat primitive cell, let $e_{m, \ell} = \min_{c \in \mathcal{C}_{m, \ell}} \mathcal{A}_{m, \ell}(c)$. For $(x_0, \xi) \in \mathbb{R}^2$, define

$$(138) \quad \mathfrak{P}_{m, \ell}^{\text{per}}(x_0, \xi) := \min_{\substack{\gamma \in W^{1,2}(x_0, x_0 + \ell) \\ \gamma(x_0) = \xi, \gamma(x_0 + \ell) = \xi - m}} \int_{x_0}^{x_0 + \ell} L_a(x, \gamma, \gamma') dx - \ell e_{m, \ell}.$$

The direct method gives the minimum. Each competitor extends to a Sobolev curve in the winding class, and its action density is ℓ -periodic. Thus its action is at least $\ell e_{m, \ell}$, independently of the chosen cut x_0 .

COROLLARY 7.5 (Vanishing of the periodic rational barrier). *For a flat primitive cell, $\mathfrak{P}_{m, \ell}^{\text{per}} \equiv 0$.*

Proof. The rational leaf through (x_0, ξ) is a full-winding minimizer and satisfies the prescribed endpoint condition. It attains $\ell e_{m,\ell}$, so the nonnegative barrier vanishes. $\square$

This is a statement about periodic winding minimizers, not about arbitrary class-A heteroclinics of the same rotation. The phase-space completion in the next section does not use this barrier or any extension of it to irrational rotation.

8. From rational flatness to complete phase-space integrability

Assume that every primitive rational cell of a is flat. For each $q = -m/\ell \in \mathbb{Q}$, Sections 6 and 7 provide a smooth invariant momentum graph $\Gamma_q = \{p = P_q(x, c)\}$ on the standard configuration torus. Every orbit on this graph is rationally periodic and class-A minimizing. We complete these rational graphs directly. No identification with an Aubry set, continuity theorem for Peierls barriers, or twist condition on the time-one map is used.

The elementary facts about increasing circle lifts and the complete Hamiltonian flow of $H = p^2/(2a)$ are proved in Appendix F.

8.1. Strict order of the rational graphs.

PROPOSITION 8.1 (Strict rational order). *If $q_1 < q_2$ are rational, then*

$$(139) \quad P_{q_1}(x, c) < P_{q_2}(x, c) \quad \text{for every } (x, c) \in \mathbb{T}^2.$$

Proof. Choose the two leaves c_1, c_2 through the same lifted point (x_0, c_0) . Their winding laws give $c_i(x) = q_i x + O(1)$ as $x \rightarrow \pm\infty$. Consequently $c_2 - c_1$ is negative for sufficiently negative x and positive for sufficiently positive x . The two-crossing principle, Lemma A.5, excludes any intersection other than x_0 . Thus $c_2 - c_1$ changes sign from negative to positive there. Their derivatives at x_0 cannot agree, since uniqueness for the Euler initial-value problem would then identify the leaves. Hence $c'_2(x_0) > c'_1(x_0)$. Multiplication by $a(x_0, c_0) > 0$ proves the claim. $\square$

8.2. *Uniform graph compactness.* Lemma D.2 proves, for every $R < \infty$,

$$(140) \quad \sup_{q \in \mathbb{Q}, |q| \leq R} (\|P_q\|_\infty + \|\partial_x P_q\|_\infty + \|\partial_c P_q\|_\infty) < \infty.$$

The estimate is independent of the primitive denominator. It follows from the velocity bound in Lemma D.1 and a Riccati estimate for the positive transverse Jacobi field of each rational foliation. In particular, rational graphs with rotations in a compact interval have a common Lipschitz constant and are precompact in $C^0(\mathbb{T}^2)$.

8.3. *One-sided limits constructed only from rational graphs.* For every $\rho \in \mathbb{R}$, define

$$(141) \quad P_\rho^-(x, c) := \sup_{q \in \mathbb{Q}, q < \rho} P_q(x, c), \quad P_\rho^+(x, c) := \inf_{q \in \mathbb{Q}, q > \rho} P_q(x, c).$$

The quantities are finite by comparison with rational graphs on either side of ρ , and $P_\rho^- \leq P_\rho^+$.

PROPOSITION 8.2 (Properties of the one-sided limits). *The functions P_ρ^-, P_ρ^+ are Lipschitz on $\mathbb{T}^2$. Their graphs are full invariant graphs, all their orbits are class-A minimizing, and the rotation number of every such orbit is ρ . If $q_j \uparrow \rho$ or $q_j \downarrow \rho$ through rational numbers, then P_{q_j} converges uniformly to the corresponding one-sided limit.*

Proof. Consider $q_j \uparrow \rho$. The estimates (140) give uniform precompactness, while monotonicity identifies every subsequential limit with the supremum in (141). The entire sequence therefore converges uniformly, and its limit has the same Lipschitz bound. The argument from above is identical.

For initial points on the approximating graphs converging to a point on the limit graph, continuous dependence of the Hamiltonian flow gives local convergence of their trajectories. Uniform convergence of the graphs then puts the limiting trajectory on the limiting graph at every time. Thus the limit graph is invariant. Convergence of momenta also gives C_{loc}^1 convergence of the configuration curves, so Lemma A.6 transfers class-A minimality.

On a full Lipschitz graph, the scalar equation $c' = P(x, c)/a(x, c)$ has unique complete solutions. Its time-one return map is an increasing degree-one lift. Uniform convergence of the graphs, with a common Lipschitz bound, implies uniform convergence of these return maps. Lemma F.1 gives convergence of their translation numbers to ρ . Finally the velocity is bounded on the limit graph, so the rotation limit along integer times is also the rotation limit along arbitrary real times. $\square$

8.4. *No jump at irrational rotations, including touching graphs.*

LEMMA 8.3 (Uniqueness of ordered irrational graphs). *Let $P_1 \leq P_2$ be full invariant Lipschitz momentum graphs for $H = p^2/(2a)$ on $\mathbb{T}^2$. If both graphs have the same irrational rotation number, then $P_1 = P_2$.*

Proof. Let $\mathcal{F}$ be the Hamiltonian time-one map on the section $x = 0$. By Lemma F.2, it is an area-preserving annulus diffeomorphism which preserves both ends. Set

$$K := \{c \in \mathbb{T} : P_1(0, c) = P_2(0, c)\}.$$

Suppose first that K is empty. The invariant graphs cannot meet at any other time section, because evolving a common point back to $x = 0$ would

give an intersection there. Hence $P_1 < P_2$ on all of $\mathbb{T}^2$. Scalar comparison for $c' = P_i(x, c)/a(x, c)$ implies that the corresponding return lifts satisfy $f_1 < f_2$ pointwise. By the strict irrational monotonicity in Lemma F.1, their translation numbers cannot be the same irrational number. This is a contradiction.

Suppose next that K is nonempty but is not all of $\mathbb{T}$. The open region

$$\Omega := \{(c, p) : P_1(0, c) < p < P_2(0, c)\}$$

has finite area, since both graphs are bounded. Each connected component is of the form

$$D_I := \{(c, p) : c \in I, P_1(0, c) < p < P_2(0, c)\},$$

where I is a component of $\mathbb{T} \setminus K$. Every D_I has strictly positive area.

Invariance of each graph and preservation of the annulus ends imply $\mathcal{F}(\Omega) = \Omega$. Thus $\mathcal{F}$ permutes its components. The two graph return maps agree on K and map K onto itself; on either graph they carry the endpoints of I to the endpoints of the interval underlying $\mathcal{F}(D_I)$. If $\mathcal{F}^n(D_I) = D_I$ for some $n \geq 1$, the corresponding orientation-preserving circle map takes I onto itself and fixes its endpoints. This supplies a periodic point on an invariant graph, forcing its rotation number to be rational by Lemma F.1. Therefore a component cannot be periodic.

It follows that $D_I, \mathcal{F}(D_I), \mathcal{F}^2(D_I), \dots$ are pairwise disjoint subsets of Ω of the same positive area. This contradicts the finite area of Ω . Hence $K = \mathbb{T}$. The section graphs agree, and invariance gives equality at every time section. $\square$

PROPOSITION 8.4 (No jump at irrational rotations). *For $\rho \notin \mathbb{Q}$, one has $P_\rho^- = P_\rho^+$.*

Proof. Apply Lemma 8.3 to the ordered invariant graphs supplied by Proposition 8.2. $\square$

8.5. *No jump at rational rotations.*

PROPOSITION 8.5 (No jump at rational rotations). *If $\rho = -m/\ell \in \mathbb{Q}$, then*

$$P_\rho^- = P_{m,\ell} = P_\rho^+.$$

Proof. Both one-sided graphs are full invariant class-A minimizing graphs of rotation $-m/\ell$. Proposition 7.4 identifies each with the smooth calibrated graph of that rational cell. Its proof uses the nonnegative quadratic calibration gap and the bounded orbits of the corrected zero-translation return map; it does not use any irrational graph or barrier theorem. $\square$

8.6. *A continuous full family and properness.*

COROLLARY 8.6 (A full minimizing graph at every rotation). *For every $\rho \in \mathbb{R}$, let $P_\rho := P_\rho^- = P_\rho^+$. Then*

$$(142) \quad \Gamma_\rho = \{(x, c, p) : p = P_\rho(x, c)\}$$

is a full invariant Lipschitz graph of class-A minimizing orbits of rotation ρ . This definition agrees with the previously constructed rational graphs. The map $\rho \mapsto P_\rho$ is strictly increasing pointwise and continuous in $C^0(\mathbb{T}^2)$.

Proof. Existence, invariance, and minimality follow from the one-sided-limit proposition and the two no-jump results. If $\rho_1 < \rho_2$, choose rationals $\rho_1 < q_1 < q_2 < \rho_2$. Then

$$P_{\rho_1} \leq P_{q_1} < P_{q_2} \leq P_{\rho_2},$$

which proves strict order. For continuity at ρ , choose rational $q^- < \rho < q^+$ approaching ρ from their respective sides. Their graphs converge uniformly to P_ρ . Every P_σ with $q^- < \sigma < q^+$ is pointwise trapped between P_{q^-} and P_{q^+} , which proves uniform continuity at ρ . $\square$

Put $M = \|a_x/a\|_\infty$ and $c_* = a_- e^{-M/2} > 0$. Lemma F.3 gives the explicit estimates

$$(143) \quad \begin{aligned} P_\rho(x, c) &\geq c_*(\rho - 1) & (\rho > 1), \\ P_\rho(x, c) &\leq -c_*(|\rho| - 1) & (\rho < -1). \end{aligned}$$

Thus the entire graph escapes uniformly in the corresponding momentum direction as $\rho \rightarrow \pm\infty$.

THEOREM 8.7 (C^0 phase-space integrability). *For every $(x, c) \in \mathbb{T}^2$, the map*

$$(144) \quad \mathbb{R} \ni \rho \longmapsto P_\rho(x, c) \in \mathbb{R}$$

is a strictly increasing homeomorphism. Consequently

$$(145) \quad \mathbb{T}^2 \times \mathbb{R}_p = \bigsqcup_{\rho \in \mathbb{R}} \Gamma_\rho.$$

Every Hamiltonian trajectory lies on exactly one such graph and is class-A minimizing.

Proof. The preceding corollary gives strict monotonicity and continuity, and (143) gives properness. Hence (144) is an increasing homeomorphism onto $\mathbb{R}$. Every initial phase point belongs to exactly one invariant graph. Uniqueness of the Hamiltonian flow identifies its trajectory with the class-A minimizing trajectory on that graph. $\square$

REMARK 8.8 (Why rational data suffice). Only rational graphs are supplied as input. Compactness constructs the one-sided graphs at every real rotation.

Rational uniqueness removes rational gaps; circle translation numbers and finite-area preservation remove irrational gaps, including the case of touching boundary graphs. No regularity in the rotation parameter is assumed.

9. Global minimizers and the absence of conjugate points

The phase-space foliation of Theorem 8.7 implies that every solution of the Euler–Lagrange equation for L_a is globally minimizing. We convert this statement to a spectral property of the Jacobi equation.

9.1. *Euler and Jacobi equations for L_a .* The Euler equation is

$$(146) \quad \frac{d}{dx}(a(x, c)c') - \frac{1}{2}a_c(x, c)c'^2 = 0.$$

Along a smooth extremal c , the second variation on $[r, s]$ is the index form

$$(147) \quad I_{c,[r,s]}[\eta] := \int_r^s (a\eta'^2 + 2a_c c' \eta \eta' + \frac{1}{2}a_{cc} c'^2 \eta^2) dx, \quad \eta \in W_0^{1,2}(r, s),$$

where all coefficients are evaluated at $(x, c(x))$. Integration by parts puts this in Sturm–Liouville form

$$(148) \quad I_{c,[r,s]}[\eta] = \int_r^s (a\eta'^2 + W_c(x)\eta^2) dx$$

for a smooth coefficient W_c . Equivalently, the linearized equation is

$$(149) \quad \mathcal{J}_c \eta = 0,$$

where $\mathcal{J}_c$ is the self-adjoint scalar Jacobi operator associated with (147). An explicit formula is recorded in Appendix G; later the drift normal form will reduce it to a simpler Newton equation.

Definition 9.1 (Conjugate points). Two points $r < s$ on an extremal c are conjugate if (149) has a nonzero solution satisfying $\eta(r) = \eta(s) = 0$.

9.2. *Minimizers have no conjugate points.*

PROPOSITION 9.2 (Every extremal is class-A minimizing). *Every solution of the Euler equation for L_a is class-A minimizing.*

Proof. An initial condition determines a phase point (x, c, p) with $p = ac'$. By Theorem 8.7, that point lies on a unique invariant graph Γ_ρ . The corresponding graph orbit is globally minimizing by Corollary 8.6; uniqueness of the Hamiltonian initial-value problem identifies it with the given solution. $\square$

THEOREM 9.3 (No conjugate points). *The Euler–Lagrange flow of L_a has no conjugate points on the universal cover.*

Proof. Let c be any extremal. Proposition 9.2 shows that the second variation is nonnegative on every compact interval. If $r < s$ were conjugate, the first Dirichlet eigenvalue of the Jacobi operator on $[r, s]$ would be at most zero. Nonnegativity forces it to equal zero. Strict Dirichlet domain monotonicity then makes the first eigenvalue negative on every properly larger interval, contradicting minimality there. The spectral argument, including the regular Sturm–Liouville reduction of (147), is proved in Appendix G. $\square$

The remaining proof of the finite-amplitude theorem uses only this no-conjugate-points conclusion, not the detailed construction of the graphs.

10. The drift normal form

We transform the quadratic metric Lagrangian L_a to a mechanical drift Lagrangian whose Jacobi equation has a velocity-independent curvature. The transformation is a smooth equivariant diffeomorphism of the configuration torus together with a positive reparametrization of the base.

10.1. *Vertical normalization.* Define

$$(150) \quad I(x) := \int_0^1 \sqrt{a(x, s)} \, ds.$$

Then I is smooth, positive, and 1-periodic. Define

$$(151) \quad F(x, c) := \frac{1}{I(x)} \int_0^c \sqrt{a(x, s)} \, ds.$$

It satisfies

$$(152) \quad \begin{aligned} F(x+1, c) &= F(x, c), \\ F(x, c+1) &= F(x, c) + 1, \\ F_c(x, c) &= \frac{\sqrt{a(x, c)}}{I(x)} > 0. \end{aligned}$$

Thus, for each x , $c \mapsto F(x, c)$ is a lift of an orientation-preserving circle diffeomorphism. Denote its inverse by

$$(153) \quad c = C(x, u), \quad F(x, C(x, u)) = u.$$

10.2. *Horizontal normalization.* Set

$$(154) \quad S := \int_0^1 I(t)^{-2} \, dt$$

and

$$(155) \quad Y(x) := \frac{1}{S} \int_0^x I(t)^{-2} \, dt.$$

Then $Y_x = I^{-2}/S > 0$ and

$$(156) \quad Y(x+1) = Y(x) + 1.$$

Let $X = Y^{-1}$.

The full configuration change is

$$(157) \quad y = Y(x), \quad u = F(x, c).$$

It descends to a smooth diffeomorphism of $\mathbb{T}^2$.

10.3. *Definition of the drift.* For $(y, u) \in \mathbb{R}^2$, put $x = X(y)$ and $c = C(x, u)$, and define

$$(158) \quad B(y, u) := SI(x)^2 F_x(x, c).$$

The deck laws of F , I , X , and C imply that B is smooth and 1-periodic in both y and u .

10.4. *Exact transformation of the action.* Let $c = c(x)$ and set

$$u(y) = F(x, c(x)), \quad y = Y(x).$$

Then

$$\frac{du}{dx} = F_x + F_c c_x, \quad \frac{dy}{dx} = \frac{I^{-2}}{S}.$$

Consequently

$$(159) \quad \begin{aligned} u_y - B(y, u) &= SI(x)^2 F_c(x, c) c_x \\ &= SI(x) \sqrt{a(x, c)} c_x. \end{aligned}$$

Since $dy = I^{-2} S^{-1} dx$, we obtain the exact identity

$$(160) \quad \int_r^s \frac{1}{2} a(x, c) c_x^2 dx = \frac{1}{S} \int_{Y(r)}^{Y(s)} \frac{1}{2} (u_y - B(y, u))^2 dy.$$

Define the drift Lagrangian

$$(161) \quad L_B(y, u, v) := \frac{1}{2} (v - B(y, u))^2.$$

THEOREM 10.1 (Drift normal form). *The change (157) has the following properties.*

- (i) *It preserves vertical deck classes and rotation numbers.*
- (ii) *It gives the action identity (160); the factor $S^{-1} > 0$ is global and harmless.*
- (iii) *It gives a bijection between fixed-endpoint competitors.*
- (iv) *It maps extremals to extremals and class-A minimizers to class-A minimizers, in both directions.*
- (v) *It preserves conjugate points, with multiplicity.*

Consequently L_B has no conjugate points.

Proof. Deck preservation follows from (152) and (156). The action identity was proved above. The configuration change is a diffeomorphism and maps endpoint data bijectively, so fixed-endpoint minimality is equivalent. First variations of the exact identity give equivalence of extremals. Differentiating a smooth one-parameter family of extremals through the diffeomorphism gives an invertible linear map between Jacobi fields; vanishing at endpoints is preserved. All coordinate computations, including the Sobolev version, are written in Appendix I. $\square$

10.5. *Euler equation in drift form.* Let $v = u_y$. Since

$$(L_B)_v = v - B, \quad (L_B)_u = -(v - B)B_u,$$

the Euler–Lagrange equation is

$$\frac{d}{dy}(v - B) + (v - B)B_u = 0.$$

Expanding the total derivative yields

$$(162) \quad u_y = v, \quad v_y = Q_B(y, u), \quad Q_B := B_y + BB_u.$$

Thus the drift disappears from the acceleration except through the scalar field Q_B .

10.6. *Jacobi equation.* Linearizing (162) along an extremal gives

$$(163) \quad \xi_{yy} = K(y, u)\xi, \quad K(y, u) := (Q_B)_u = (B_y + BB_u)_u.$$

The decisive feature is that K depends only on (y, u) and not on the velocity v . This is the form required by the Hopf slab argument.

11. Hopf rigidity for the drift Lagrangian

Following the Green–Hopf strategy and its Newton-equation variants [6], [5], [4], and [3], we now prove that a periodic drift Lagrangian

$$L_B(y, u, v) = \frac{1}{2}(v - B(y, u))^2$$

without conjugate points must satisfy $B_u = 0$. This theorem is independent of the preceding cell construction.

11.1. *The drift flow.* The Euler equation is the complete first-order system

$$(164) \quad \dot{y} = 1, \quad \dot{u} = v, \quad \dot{v} = Q_B(y, u),$$

where $Q_B = B_y + BB_u$. Since Q_B is smooth and periodic, the flow is complete on

$$\mathcal{P}_B := \mathbb{T}_y \times \mathbb{T}_u \times \mathbb{R}_v.$$

Its vector field is

$$(165) \quad D := \partial_y + v\partial_u + Q_B(y, u)\partial_v.$$

The standard volume form

$$(166) \quad d\lambda = dy \, du \, dv$$

is invariant because

$$\operatorname{div} D = 0.$$

The linearized scalar equation is

$$(167) \quad J''(t) = K(y(t), u(t))J(t), \quad K = (Q_B)_u.$$

By Theorem 10.1, there are no conjugate points along any orbit.

11.2. Finite-horizon Green slopes. Fix $z \in \mathcal{P}_B$ and parametrize its orbit so that z occurs at time zero. For $T > 0$, let $J_{T,z}$ be the solution of (167) satisfying

$$(168) \quad J_{T,z}(-T) = 0, \quad J_{T,z}(0) = 1.$$

No conjugate points implies

$$J_{T,z}(t) > 0 \quad (-T < t \leq 0).$$

Define

$$(169) \quad r_T(z) := J'_{T,z}(0).$$

LEMMA 11.1 (Monotonicity of the Green slopes). *If $S > T > 0$, then*

$$r_S(z) < r_T(z).$$

Proof. The Wronskian of $J_{S,z}$ and $J_{T,z}$ is constant. Evaluating at zero and at $-T$ gives

$$r_T - r_S = J_{S,z}(-T)J'_{T,z}(-T).$$

Both factors are positive: the first by no conjugate points, the second because $J_{T,z}$ leaves its simple zero into positive values. Hence the difference is positive. $\square$

LEMMA 11.2 (Uniform Riccati bound). *There exists $C_K < \infty$, depending only on $\|K\|_{L^\infty(\mathbb{T}^2)}$, such that*

$$|r_T(z)| \leq C_K \quad \text{for all } z \in \mathcal{P}_B, \quad T \geq 1.$$

Proof. Set $w = J'/J$ on $(-T, 0]$. It solves

$$w' + w^2 - K = 0.$$

Scalar Riccati comparison with the constant equations having coefficients $\pm\|K\|_\infty$ gives a lower bound independent of T and, because the past zero is

at distance at least one, a uniform upper bound at time zero. The explicit comparison and the case $K = 0$ are in Appendix H. $\square$

By monotonicity and boundedness the limit

$$(170) \quad r(z) := \lim_{T \rightarrow +\infty} r_T(z)$$

exists and is bounded.

PROPOSITION 11.3 (Global Green–Riccati field). *The function $r : \mathcal{P}_B \rightarrow \mathbb{R}$ is bounded and Borel measurable. Along every flow orbit it is continuously differentiable and satisfies*

$$(171) \quad Dr + r^2 - K = 0.$$

The identity also holds in the distributional sense on $\mathcal{P}_B$.

Proof. For finite T , continuous dependence for the Jacobi equation makes r_T continuous on the region where it is defined; the monotone pointwise limit is Borel. On every compact orbit interval, normalized Jacobi fields have uniform C^1 bounds and converge to a positive global Jacobi field. Its logarithmic derivative is the flow translate of r and obeys the Riccati equation. Flow boxes and Fubini convert the orbitwise equality into the distributional one. Appendix H gives the full measure-theoretic argument. $\square$

11.3. Integration over velocity slabs. For $R > 0$, set

$$(172) \quad A(R) := \int_{\mathbb{T}^2 \times [-R, R]} r(y, u, v)^2 dy du dv.$$

The function A is finite, nondecreasing, and locally absolutely continuous. For almost every R ,

$$(173) \quad A'(R) = \int_{\mathbb{T}^2} (r(y, u, R)^2 + r(y, u, -R)^2) dy du.$$

PROPOSITION 11.4 (Velocity-slab identity). *For almost every $R > 0$,*

$$(174) \quad \begin{aligned} A(R) = & \int_{\mathbb{T}^2} Q_B(y, u) r(y, u, -R) dy du \\ & - \int_{\mathbb{T}^2} Q_B(y, u) r(y, u, R) dy du. \end{aligned}$$

Proof. Integrate the distributional identity $Dr = K - r^2$ over the slab. Since D is divergence-free, the y and u boundary terms cancel by periodicity, while the flux through the two velocity faces is

$$\int_{v=R} Q_B r - \int_{v=-R} Q_B r.$$

The integral of $K = (Q_B)_u$ over the slab vanishes:

$$\int_{\mathbb{T}^2 \times [-R, R]} K = 2R \int_{\mathbb{T}^2} (Q_B)_u dy du = 0.$$

Rearranging gives (174). Since r need not be smooth, the rigorous proof first tests with smooth cutoffs in v and passes to characteristic functions at Lebesgue radii; see Appendix H. $\square$

COROLLARY 11.5 (Hopf differential inequality). *For almost every $R > 0$,*

$$(175) \quad A(R)^2 \leq 2\|Q_B\|_{L^2(\mathbb{T}^2)}^2 A'(R).$$

Proof. By (174) and Cauchy–Schwarz,

$$\begin{aligned} A(R) &\leq \|Q_B\|_2 (\|r(\cdot, -R)\|_2 + \|r(\cdot, R)\|_2) \\ &\leq \sqrt{2}\|Q_B\|_2 \sqrt{A'(R)}. \end{aligned}$$

Square the inequality. $\square$

11.4. Vanishing of the curvature.

THEOREM 11.6 (Hopf rigidity for the drift equation). *If L_B has no conjugate points, then*

$$K = (B_y + BB_u)_u \equiv 0.$$

Proof. If $Q_B = 0$, the slab identity immediately gives $A = 0$. Otherwise put $C = 2\|Q_B\|_2^2$. If $A(R_0) > 0$, then for almost every $R \geq R_0$,

$$A'(R) \geq \frac{A(R)^2}{C}.$$

Since A remains positive,

$$\left(\frac{1}{A(R)} \right)' = -\frac{A'(R)}{A(R)^2} \leq -\frac{1}{C}.$$

After finite integration the positive function $1/A$ would become nonpositive, a contradiction. Thus $A(R) = 0$ for every R , so $r = 0$ almost everywhere. The Riccati equation gives $K = 0$ in distributions and hence pointwise by smoothness. $\square$

11.5. From curvature zero to drift rigidity. Let

$$(176) \quad w(y, u) := B_u(y, u).$$

The equation $K = 0$ is

$$(177) \quad w_y + Bw_u + w^2 = 0.$$

PROPOSITION 11.7 (Drift rigidity). *A smooth periodic solution of (177) is identically zero. Hence $B_u \equiv 0$.*

Proof. Along a characteristic of

$$\dot{y} = 1, \quad \dot{u} = B(y, u),$$

one has

$$\frac{d}{dy}w(y, u(y)) = -w(y, u(y))^2.$$

If the value at one point were positive, the explicit scalar solution would blow up in finite backward time; if negative, it would blow up in finite forward time. The characteristic flow is complete and w is smooth and bounded on the torus. Thus every value is zero. $\square$

Combining Theorem 11.6 and Proposition 11.7 yields the standalone conclusion

$$(178) \quad L_B \text{ has no conjugate points} \implies B_u = 0.$$

12. From drift rigidity to multiplicative separability

We reverse the normal form and complete the proof of the finite-amplitude cell theorem.

12.1. *Differentiating the drift.* Recall

$$B(y, u) = SI(x)^2 F_x(x, c), \quad x = X(y), \quad c = C(x, u),$$

and

$$F_c(x, c) = \frac{\sqrt{a(x, c)}}{I(x)}.$$

At fixed y , the variable $x = X(y)$ is fixed. Differentiating the inverse identity $F(x, C(x, u)) = u$ gives

$$(179) \quad C_u(x, u) = \frac{1}{F_c(x, C(x, u))}.$$

Therefore

$$(180) \quad \begin{aligned} B_u(y, u) &= SI(x)^2 F_{xc}(x, c) C_u(x, u) \\ &= SI(x)^2 \frac{F_{xc}(x, c)}{F_c(x, c)}. \end{aligned}$$

All factors outside F_{xc} are positive. Hence

$$(181) \quad B_u \equiv 0 \iff F_{xc} \equiv 0.$$

12.2. *Separation of the coefficient.* If $F_{xc} = 0$, then $F_c(x, c)$ is independent of x . There is a smooth positive 1-periodic function G such that

$$(182) \quad F_c(x, c) = G(c).$$

Using the definition of F_c ,

$$(183) \quad \frac{\sqrt{a(x, c)}}{I(x)} = G(c).$$

Consequently

$$(184) \quad a(x, c) = I(x)^2 G(c)^2.$$

The normalization of F gives

$$(185) \quad \int_0^1 G(c) dc = 1.$$

Thus $A = I^2$ and $B = G^2$ provide the required multiplicative separation.

Conversely, Section 5 proved that every multiplicatively separable coefficient has flat rational cells.

Proof of Theorem 1.4. The implications

$$(II) \iff (III) \iff (IV)$$

are elementary: take logarithms and integrate the vanishing mixed derivative on the torus.

The implication $(II) \Rightarrow (I)$ is Proposition 5.2.

Assume (I). Flatness creates rational minimizing foliations by Theorem 6.7, and their Hamiltonian calibrations make all rational leaves class-A minimizing by Corollary 7.3. Section 8 completes these graphs to the phase-space foliation (145). Hence every extremal is minimizing and the flow has no conjugate points by Theorem 9.3. The drift normal form preserves conjugate points, so L_B has none. Hopf rigidity and Proposition 11.7 give $B_u = 0$. Equations (180)–(184) then prove $a(x, c) = A(x)B(c)$. Thus $(I) \Rightarrow (II)$. $\square$

12.3. *Equivalent and contrapositive forms.*

COROLLARY 12.1 (Normalized separability). *Under flat rational cells, the coefficient has the canonical representation*

$$a(x, c) = I(x)^2 G(c)^2, \quad I(x) = \int_0^1 \sqrt{a(x, s)} ds, \quad \int_0^1 G = 1.$$

The functions I and G are uniquely determined.

Proof. Existence was proved above. The formula for I follows by integrating $\sqrt{a} = IG$ in c and using $\int G = 1$. Then $G = \sqrt{a}/I$. $\square$

COROLLARY 12.2 (Finite-amplitude detection). *If $(\log a)_{xc}$ is not identically zero, then there is a primitive pair (m, ℓ) and two phases $Y_1, Y_2 \in \mathbb{T}_\ell$ such that*

$$E_{m,\ell}(Y_1) \neq E_{m,\ell}(Y_2).$$

Proof. This is the contrapositive of Theorem 1.4. $\square$

Remark 12.3 (Why all primitive rational cells appear). The proof uses rational flatness to obtain a graph in every rational rotation class and then to complete phase space. The theorem does not claim that a single flat cell, or any fixed finite set of cells, forces separability. The complete rational family is the global finite-amplitude datum.

13. Return to the original potential

We now return from the independent quadratic theorem to the original Euler–Lagrange equation. At this point every use of the extra holonomy condition has already occurred; we keep the final calculation explicit.

13.1. *Flatness of the reduced coefficient.* Let U_0 be the normalized zero-rotation foliation and set

$$a(x, c) = U_{0,c}(x, c)^2.$$

The identity $\mathfrak{h}_{0,1} = \text{Id}$ makes this a smooth positive function on the product torus. For each primitive rational pair (m, ℓ) , the identity $\mathfrak{h}_{m,\ell} = \text{Id}$ makes every transformed leaf a closed curve in the winding class $c(x + \ell) = c(x) - m$. The two-calibration comparison in Proposition 3.8, followed by the surjectivity of the transformed phase map, proves that the corresponding cell energy is independent of its gauge phase. Hence every primitive cell of a is flat.

The finite-amplitude theorem now applies. There are smooth positive periodic functions A and B such that

$$(186) \quad U_{0,c}(x, c)^2 = A(x)B(c).$$

Since $U_{0,c} > 0$, we take the positive square roots and write

$$(187) \quad U_{0,c}(x, c) = \alpha(x)\beta(c), \quad \alpha = \sqrt{A}, \quad \beta = \sqrt{B}.$$

13.2. *The vertical deck law removes the spatial factor.* The vertical deck identity is

$$U_0(x, c + 1) - U_0(x, c) = 1.$$

Using (187),

$$\begin{aligned}
 (188) \quad 1 &= \int_c^{c+1} U_{0,c}(x, s) ds \\
 &= \alpha(x) \int_0^1 \beta(s) ds.
 \end{aligned}$$

The second factor is a positive constant, so $\alpha(x)$ is constant. Absorb that constant into β and define

$$H(c) := \int_0^c \beta(s) ds.$$

Then $H' > 0$ and (188) gives

$$(189) \quad H(c+1) = H(c) + 1.$$

Because $U_{0,c} = H'(c)$,

$$\partial_c(U_0(x, c) - H(c)) = 0.$$

Thus there is a smooth function ψ such that

$$(190) \quad U_0(x, c) = \psi(x) + H(c).$$

The identity zero holonomy $U_0(x+1, c) = U_0(x, c)$ makes ψ 1-periodic.

13.3. *The force is independent of the vertical variable.* Each zero leaf solves the original equation. Substitution of (190) gives

$$(191) \quad \psi''(x) = -V_u(x, \psi(x) + H(c)) \quad \text{for every } (x, c) \in \mathbb{R}^2.$$

For fixed x , the map

$$c \mapsto \psi(x) + H(c)$$

is strictly increasing, commutes with integer translation, and is onto $\mathbb{R}$. Therefore (191) implies

$$(192) \quad V_u(x, u) = -\psi''(x) \quad \text{for every } (x, u) \in \mathbb{R}^2.$$

Finally, V is 1-periodic in u , so

$$0 = \int_0^1 V_u(x, u) du = -\psi''(x).$$

Hence

$$V_u \equiv 0.$$

Proof of Theorem 1.2. Appendix B gives the transverse smoothness and positivity of every normalized identity-holonomy rational foliation. The zero foliation gives the product-torus coefficient $a = U_{0,c}^2$ by Proposition 3.6. Every other rational foliation makes the corresponding gauge-fixed cell flat by Proposition 3.10. The finite-amplitude Theorem 1.4 gives (186). The vertical deck

calculation (188) gives the additive form (190), and the zero-leaf equation then gives $V_u \equiv 0$. $\square$

13.4. *Exact accounting of the extra assumption.* For clarity, the proof used identity primitive holonomy in exactly three ways, the first two being the substantive ones.

- (i) $\mathfrak{h}_{0,1} = \text{Id}$ makes U_0 periodic in x and therefore makes $a = U_{0,c}^2$ a coefficient on the product torus rather than a quadratic density on a mapping torus.
- (ii) $\mathfrak{h}_{m,\ell} = \text{Id}$ makes each nonzero rational leaf close under the primitive deck transformation, so its zero-coordinate image belongs to one fixed rational winding class and represents one gauge phase.
- (iii) The same rigid deck laws make the transverse finite differences periodic in x , which is used in Appendix B to derive smooth positive transverse Jacobi fields from the originally continuous leaf parameter.

After Proposition 3.10, the proof is an independent theorem about the periodic quadratic coefficient a and no holonomy map appears again.

Remark 13.1 (Scope). The theorem assumes one identity-holonomy foliation for each primitive rational class. It assumes no irrational foliation and no regularity in the rotation parameter. Section 2 shows why the holonomy qualifier cannot be deleted: the ordinary all-rotation statement is false even for the autonomous potential (22).

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Throughout this process, ChatGPT was used as a research and writing tool rather than as an autonomous researcher. The author remained actively involved in the development of the mathematics, independently evaluated and reconstructed AI-assisted arguments, made all decisions about which arguments and formulations to retain, and verified the mathematical content of the final manuscript.

In particular, ChatGPT contributed to the development and refinement of the explicit counterexample in Section 2, including the square-completion calibration, the construction of rotating and zero-rotation minimizing foliations, and the analysis of the nontrivial zero-rotation holonomy. It also assisted in identifying and clarifying the distinction between ordinary deck-invariant

foliations and the identity primitive holonomy hypothesis used in the positive rigidity theorem.

ChatGPT contributed to the zero-foliation reduction in Section 3, including the calibration identity reducing the original action to a periodic quadratic Lagrangian, the treatment of the deck laws, and the transfer to phase-flat rational cells. It also assisted with the formulation of the gauge-fixed resonant cell problem in Section 4, the construction of ordered minimizing foliations from flat cells in Section 6, and the Hamiltonian calibration and rational-uniqueness arguments in Section 7.

A further contribution concerned the passage from rational invariant graphs to a complete continuous foliation of phase space in Section 8. ChatGPT assisted with developing and refining the arguments based on order, compactness, circle translation numbers, and area preservation, including the one-sided limit construction in Proposition 8.2, the treatment of touching irrational graphs in Lemma 8.3, and the phase-space completion and global minimality statement in Theorem 8.7. It also assisted with the subsequent absence-of-conjugate-points argument in Section 9.

ChatGPT contributed to the development and refinement of the drift normal form in Section 10 and the Green–Hopf argument in Section 11 and Appendix H. This included the construction of the Green–Riccati field, the weak Riccati equation, integration over velocity slabs, the boundary-flux calculation, and the differential inequality used to establish drift rigidity. It also assisted with the deduction of multiplicative separability in Section 12 and the return to the original potential in Section 13.

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All mathematical statements and proofs retained in the manuscript, and all citations and references, were independently checked by the author. The author takes full responsibility for the correctness and final content of the work, including all material developed with the assistance of generative AI.

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Appendix A. Sobolev competitors, traces, and closure of minimizers

This appendix proves the variational facts used throughout the paper. We state them for the original action. The same Sobolev approximation and closure arguments apply to the two later Lagrangians L_a and L_B : all three have at most quadratic growth in velocity with smooth periodic coefficients. No assertion about arbitrary superquadratic Tonelli Lagrangians on $W^{1,2}$ is intended.

A.1. Sobolev spaces and endpoint traces. For a bounded interval $I = (a, b)$, the Sobolev space $W^{1,2}(I)$ consists of L^2 functions whose weak derivative belongs to L^2 . In one dimension, every element has an absolutely continuous representative satisfying

$$f(x) - f(y) = \int_y^x f'(t) dt.$$

In particular, endpoint values are well-defined and the embedding $W^{1,2}(I) \hookrightarrow C^0(\bar{I})$ is continuous.

The local space $W_{\text{loc}}^{1,2}(\mathbb{R})$ consists of functions whose restriction to every compact interval belongs to $W^{1,2}$. The space $W_0^{1,2}(a, b)$ is the closure of $C_c^\infty(a, b)$ in $W^{1,2}$ and satisfies

$$(193) \quad W_0^{1,2}(a, b) = \{f \in W^{1,2}(a, b) : f(a) = f(b) = 0\}.$$

A.2. Compact variations and fixed endpoint competitors.

LEMMA A.1 (Equivalence of competitor classes). *Let $u \in W^{1,2}(a, b)$. The following are equivalent.*

- (i) $\mathcal{A}_V[u; a, b] \leq \mathcal{A}_V[u + \varphi; a, b]$ for every $\varphi \in C_c^\infty(a, b)$.
- (ii) The same inequality holds for every $\varphi \in W_0^{1,2}(a, b)$.

- (iii) $\mathcal{A}_V[u; a, b] \leq \mathcal{A}_V[v; a, b]$ for every $v \in W^{1,2}(a, b)$ satisfying $v(a) = u(a)$ and $v(b) = u(b)$.

Proof. The equivalence of (ii) and (iii) follows from (193). Clearly (ii) implies (i). Assume (i) and let $\varphi \in W_0^{1,2}(a, b)$. Choose $\varphi_j \in C_c^\infty(a, b)$ with $\varphi_j \rightarrow \varphi$ in $W^{1,2}$. Then $u' + \varphi_j' \rightarrow u' + \varphi'$ in L^2 , so the kinetic terms converge in L^1 . Sobolev embedding gives uniform convergence $u + \varphi_j \rightarrow u + \varphi$. Since V is smooth and periodic, it is bounded and uniformly continuous, hence the potential terms converge in L^1 . Passing to the limit proves (ii). $\square$

COROLLARY A.2 (Equivalent definition of class-A minimality). *A function is a class-A minimizer if and only if it minimizes on every compact interval against all competitors with the same endpoint trace.*

A.3. Euler–Lagrange regularity.

LEMMA A.3 (Weak equation and smoothness). *A local minimizer of $\mathcal{A}_V$ is a smooth solution of $u'' = -V_u(x, u)$.*

Proof. For $\psi \in C_c^\infty(a, b)$, apply minimality to $u + \varepsilon\psi$ for positive and negative ε . Differentiation at zero gives

$$\int_a^b (u'\psi' - V_u(x, u)\psi) dx = 0.$$

Thus $u'' = -V_u(x, u)$ in distributions. Since u is continuous and the right side is continuous, $u \in C^2$. Repeated differentiation of the ODE gives C^∞ regularity. $\square$

A.4. Splicing.

LEMMA A.4 (Sobolev splicing). *Let $u, v \in W^{1,2}(a_-, a_+)$ and suppose $u(a) = v(a)$, $u(b) = v(b)$ for $a_- < a < b < a_+$. The functions obtained by interchanging the two arcs on $[a, b]$ belong to $W^{1,2}(a_-, a_+)$. The sum of their actions equals the sum of the actions of u and v .*

Proof. The matching trace values make the piecewise definitions continuous. A continuous function which is $W^{1,2}$ on finitely many adjacent intervals belongs to $W^{1,2}$ globally, with weak derivative equal to the piecewise derivative. The action sum identity is pointwise away from the endpoints, which have measure zero. $\square$

LEMMA A.5 (Two-crossing principle). *Two distinct class-A minimizers of the same one-dimensional Tonelli Lagrangian cannot agree at two distinct points.*

Proof. Interchange the arcs between two intersections. By Lemma A.4, the sum of the two competitor actions is unchanged. Minimality forces each spliced curve to be a minimizer. It is therefore smooth. Smoothness at a splice point forces the original derivatives to agree there. The two original solutions have the same Cauchy data, so ODE uniqueness identifies them. $\square$

A.5. Closure under local convergence.

LEMMA A.6 (Closure of class-A minimizers). *Fix one of the Lagrangians L_V , L_a , or L_B used in this paper, and let u_j be class-A minimizers for it. If $u_j \rightarrow u$ in C_{loc}^1 , then u is class-A minimizing. The same conclusion holds if $u_j \rightarrow u$ in C_{loc}^0 and the second derivatives of u_j are uniformly bounded on compact intervals.*

Proof. The second hypothesis implies C_{loc}^1 convergence: bounded second derivatives give precompactness of the first derivatives, and every subsequential derivative limit differentiates the fixed C^0 limit.

Fix $[a, b]$ and a competitor v with the endpoint trace of u . Let ℓ_j be the affine function whose endpoint values are $u_j(a) - u(a)$ and $u_j(b) - u(b)$, and set $v_j = v + \ell_j$. Then v_j has the endpoint trace of u_j and $v_j \rightarrow v$ in $W^{1,2}$. Minimality gives

$$\mathcal{A}[u_j; a, b] \leq \mathcal{A}[v_j; a, b].$$

Continuity of the action under these convergences yields the result. $\square$

A.6. Second variation.

LEMMA A.7 (Nonnegative index form). *Let u be a class-A minimizer of the original action. Then*

$$(194) \quad \int_a^b (\eta'^2 + Q_u(x, u)\eta^2) \, dx \geq 0$$

for every $\eta \in W_0^{1,2}(a, b)$. For a general smooth Tonelli Lagrangian, its standard index form is nonnegative on every compact interval.

Proof. Differentiate the action twice along $u + \varepsilon\eta$ for smooth compactly supported η , use that the first variation vanishes, and then pass to $W_0^{1,2}$ by density. The general formula follows identically from the second derivative of the Lagrangian. $\square$

A.7. *Periodic cell competitors and local minimality.* We record a standard localization fact which is occasionally useful when a periodic minimizer is first obtained from a cell problem.

LEMMA A.8 (Tiling a cell competitor). *Let L be a smooth periodic Tonelli Lagrangian. Suppose a rationally periodic extremal minimizes the action on*

every multiple of its primitive cell among curves with the same winding and endpoint data. Then it is class-A minimizing.

Proof. A compactly supported competitor is contained in a sufficiently long multiple of the primitive cell and agrees with the original extremal at the endpoints of that multiple. It is therefore an admissible fixed-endpoint competitor there. The cell inequality gives the local inequality. In the main text we obtain the stronger conclusion by Hamiltonian calibration, so no collar error is needed. $\square$

Appendix B. Automatic transverse regularity of a rational foliation

We prove Proposition 3.2. The argument uses no regularity of the foliation in its given leaf label.

B.1. *Canonical normalization.* Let $\widetilde{U}(x, t)$ be a rational foliation of rotation $-m/\ell$. The map $h(t) = \widetilde{U}(0, t)$ is an increasing lift of a circle homeomorphism. Reparametrize the leaves by

$$(195) \quad U(x, c) := \widetilde{U}(x, h^{-1}(c)).$$

Then

$$(196) \quad \begin{aligned} U(0, c) &= c, \\ U(x, c+1) &= U(x, c) + 1, \\ U(x+\ell, c) &= U(x, c) - m. \end{aligned}$$

For every c , $U(\cdot, c)$ solves

$$U_{xx} = Q(x, U).$$

B.2. *Positive finite differences.* For $s \neq 0$, define

$$(197) \quad \phi_s(x, c) := \frac{U(x, c+s) - U(x, c)}{s}.$$

Strict ordering of the leaves gives $\phi_s > 0$ for both signs of s , and normalization gives

$$(198) \quad \phi_s(0, c) = 1.$$

The deck laws show that ϕ_s is ℓ -periodic in x .

Subtracting the two equations and using the integral mean-value formula gives

$$(199) \quad \partial_{xx}\phi_s = a_s(x, c)\phi_s,$$

where

$$(200) \quad a_s(x, c) := \int_0^1 Q_u(x, U(x, c) + \theta(U(x, c+s) - U(x, c))) d\theta.$$

Thus

$$(201) \quad |a_s| \leq M := \|Q_u\|_{L^\infty(\mathbb{T}^2)}.$$

B.3. *Uniform logarithmic derivative bound.*

LEMMA B.1 (Periodic Riccati estimate). *Let $\phi > 0$ be an entire solution of $\phi'' = a(x)\phi$ with $|a| \leq M$. Then*

$$(202) \quad |\phi'(x)| \leq \sqrt{M} \phi(x)$$

for every x , with the interpretation $\phi' = 0$ when $M = 0$.

Proof. Set $z = \phi'/\phi$. Then $z' = a - z^2$. If $z(x_0) > \sqrt{M}$, comparison backward with $w' = M - w^2$ forces a finite backward pole. If $z(x_0) < -\sqrt{M}$, forward comparison forces a finite forward pole. Either pole would give a zero or singularity of the positive entire solution. Hence $|z| \leq \sqrt{M}$. $\square$

Applied to (199), this gives

$$(203) \quad e^{-\sqrt{M}|x|} \leq \phi_s(x, c) \leq e^{\sqrt{M}|x|}$$

when measured from $x = 0$. On one period, ϕ_s , ϕ'_s , and ϕ''_s are uniformly bounded, independently of small s and locally uniformly in c .

B.4. *Existence and uniqueness of the transverse derivative.* Arzela–Ascoli gives, along every sequence $s_j \rightarrow 0$, a subsequence with

$$\phi_{s_j}(\cdot, c) \rightarrow \phi(\cdot, c) \quad \text{in } C^1([0, \ell]).$$

The coefficients a_{s_j} converge uniformly to $Q_u(x, U(x, c))$, so

$$(204) \quad \phi_{xx} = Q_u(x, U(x, c))\phi, \quad \phi(0, c) = 1, \quad \phi > 0,$$

and ϕ is ℓ -periodic.

LEMMA B.2 (Uniqueness of the normalized positive periodic Jacobi field). *Equation (204) has at most one positive ℓ -periodic solution satisfying $\phi(0) = 1$.*

Proof. Let ϕ_1, ϕ_2 be two such solutions. Their Wronskian W is constant, and

$$\left(\frac{\phi_1}{\phi_2} \right)' = \frac{W}{\phi_2^2}.$$

If $W \neq 0$, the ratio is strictly monotone and cannot be periodic. Thus $W = 0$, the ratio is constant, and normalization makes it one. $\square$

Every subsequential limit is therefore the same. Hence the full family converges and

$$(205) \quad U_c(x, c) := \lim_{s \rightarrow 0} \phi_s(x, c)$$

exists. It is positive, periodic, and solves the Jacobi equation.

B.5. *Joint continuity.* Let $(c_j, s_j) \rightarrow (c, 0)$. The same uniform estimates apply to $\phi_{s_j}(\cdot, c_j)$. Every limit solves the Jacobi equation at c and is normalized at zero, so uniqueness identifies it with $U_c(\cdot, c)$. Thus U_c is jointly continuous. The convergence of derivatives in x shows that U_{xc} is jointly continuous as well.

B.6. *Smooth bootstrap by a Dirichlet implicit-function argument.* The positive Jacobi field already constructed removes the Dirichlet kernel. This gives an explicit invertibility statement, rather than inferring higher derivatives from uniqueness of a periodic solution.

LEMMA B.3 (Dirichlet nondegeneracy and smooth dependence). *Let $L(x, u, v)$ be smooth with $L_{vv} > 0$. Suppose that a family of smooth extremals $U(\cdot, t)$ on $[0, \ell]$ is continuous in C^1 and satisfies*

$$U(0, t) = t, \quad U(\ell, t) = t - m.$$

Suppose that, along each member of this family, the Jacobi equation has a smooth solution ϕ which is strictly positive on $[0, \ell]$. Then U is jointly smooth in (x, t) .

Proof. Fix t_0 and evaluate the coefficients along $U(\cdot, t_0)$. Its Jacobi operator is

$$J\eta = -(A\eta')' + W\eta, \quad A = L_{vv} > 0, \quad W = L_{uu} - \frac{d}{dx}L_{uv}.$$

Since $J\phi = 0$ and $\phi > 0$, integration by parts gives the ground-state identity

$$(206) \quad \int_0^\ell (A\eta'^2 + W\eta^2) dx = \int_0^\ell A\phi^2 \left(\left(\frac{\eta}{\phi} \right)' \right)^2 dx \quad (\eta \in W_0^{1,2}(0, \ell)).$$

The boundary term vanishes because η has zero endpoint values. A Dirichlet solution of $J\eta = 0$ therefore has η/ϕ constant, and its endpoint values force that constant to be zero. Thus the Dirichlet kernel is trivial.

Fix $0 < \beta < 1$ and let

$$X = \{w \in C^{2,\beta}([0, \ell]) : w(0) = w(\ell) = 0\}.$$

The regular scalar Dirichlet operator is consequently an isomorphism $J : X \rightarrow C^{0,\beta}([0, \ell])$: the scalar boundary-value problem has a unique solution by its nonvanishing Dirichlet determinant, and its Green operator has the usual $C^{0,\beta}$ to $C^{2,\beta}$ bound. For $u = t - mx/\ell + w$, define the smooth map

$$\mathcal{F}(t, w) = L_u(x, u, u') - \frac{d}{dx}L_v(x, u, u') \quad \text{from } \mathbb{R} \times X \text{ to } C^{0,\beta}([0, \ell]).$$

At the reference extremal its derivative in w is J . The Banach-space implicit-function theorem therefore supplies a unique smooth local branch $w = w(t)$ of solutions with the prescribed endpoints.

The given family agrees with this branch. Indeed, solving its Euler equation for U_{xx} expresses that derivative as a smooth function of (x, U, U_x) . Its assumed

C^1 continuity therefore gives C^2 continuity, and differentiation in x gives locally uniform C^3 bounds. Interpolation gives continuity in $C^{2,\beta}$ for $0 < \beta < 1$. Thus the given family lies in the uniqueness neighborhood of the implicit-function theorem for t near t_0 . Smooth dependence in $C^{2,\beta}$, followed by differentiation of the Euler equation, gives every mixed derivative in x and t . As t_0 was arbitrary, the family is jointly smooth. $\square$

For the original Lagrangian L_V , one has $A = 1$ and $W = Q_u(x, U)$. The preceding subsections give a positive periodic Jacobi field $\phi = U_c$. The uniform bounds on ϕ_s and ϕ'_s show directly that $\|U(\cdot, c+s) - U(\cdot, c)\|_{C^1([0,\ell])} \leq C|s|$ locally in c , giving the required C^1 continuity. Lemma B.3 therefore applies with $t = c$. The winding deck law extends the conclusion from $[0, \ell]$ to the whole lifted plane.

PROPOSITION B.4 (Transverse regularity). *The normalized rational foliation U belongs to $C^\infty(\mathbb{R}^2)$ and $U_c > 0$. The derivative U_c is ℓ -periodic in x and 1-periodic in c .*

Proof. Existence, positivity, and smoothness were proved above. Differentiate the deck laws (196) to obtain periodicity. $\square$

Appendix C. The direct method and the constrained cell equation

We prove Proposition 4.3 and the quantitative facts used in Section 4.

C.1. Coercivity in the corrector gauge. Fix a primitive pair (m, ℓ) , a lift $Y \in \mathbb{R}$, and define on $W_{\text{per},0}^{1,2}(0, \ell)$

$$(207) \quad \mathcal{F}_Y(\varphi) := \frac{1}{2\ell^3} \int_0^\ell a\left(x, \frac{Y + \varphi(x) - mx}{\ell}\right) (\varphi'(x) - m)^2 dx.$$

By the elementary inequality

$$|\varphi'|^2 \leq 2|\varphi' - m|^2 + 2m^2$$

and $a \geq a_-$,

$$(208) \quad \|\varphi'\|_{L^2}^2 \leq \frac{4\ell^3}{a_-} \mathcal{F}_Y(\varphi) + 2m^2\ell.$$

Poincaré's inequality on the mean-zero periodic space gives

$$(209) \quad \|\varphi\|_{L^2} \leq C_\ell \|\varphi'\|_{L^2}.$$

Thus every energy-bounded sequence is bounded in $W^{1,2}$.

C.2. *Lower semicontinuity.* Let $\varphi_j \rightharpoonup \varphi$ weakly in $W_{\text{per},0}^{1,2}$. After passage to a subsequence, compact one-dimensional Sobolev embedding gives uniform convergence. Hence

$$a\left(x, \frac{Y + \varphi_j - mx}{\ell}\right) \longrightarrow a\left(x, \frac{Y + \varphi - mx}{\ell}\right)$$

uniformly. Write the integrand as

$$\left| \sqrt{a\left(x, \frac{Y + \varphi_j - mx}{\ell}\right)} (\varphi_j' - m) \right|^2.$$

The coefficient converges strongly in L^∞ , while $\varphi_j' - m \rightharpoonup \varphi' - m$ in L^2 . Weak lower semicontinuity of the L^2 norm gives

$$(210) \quad \mathcal{F}_Y(\varphi) \leq \liminf_j \mathcal{F}_Y(\varphi_j).$$

The direct method now yields a minimizer.

C.3. *First variation.* Let $c = (Y + \varphi - mx)/\ell$. A corrector variation $\varphi + \varepsilon\eta$ with $\eta \in W_{\text{per},0}^{1,2}$ induces $\delta c = \eta/\ell$. Differentiation gives

$$(211) \quad 0 = \int_0^\ell \left(a(x, c) c' \eta' + \frac{1}{2} a_c(x, c) c'^2 \eta \right) dx$$

for every periodic zero-mean η , after multiplication by an irrelevant positive constant.

Define the distribution

$$R := -\frac{d}{dx}(a(x, c)c') + \frac{1}{2}a_c(x, c)c'^2.$$

Equation (211) says $\langle R, \eta \rangle = 0$ for every periodic zero-mean test function. The annihilator of the zero-mean subspace consists of constants. Hence

$$R = \lambda$$

for some constant λ , proving (102).

C.4. *Regularity.* Set

$$p(x) := a(x, c(x))c'(x).$$

The weak constrained equation gives

$$(212) \quad p' = \frac{1}{2}a_c(x, c)c'^2 - \lambda \quad \text{in distributions.}$$

Since $c' \in L^2$ and a_c is bounded, the right side belongs to L^1 . Hence $p \in W^{1,1}(0, \ell)$ and has a continuous, bounded representative. Because $a \geq a_- > 0$,

$$c' = \frac{p}{a(x, c)}$$

is continuous. Therefore $c \in C^1$. Equation (212) now has a continuous right side, so $p \in C^1$ and $c \in C^2$. Repeated differentiation gives $c \in C^\infty$.

Expanding once gives the formula used later:

$$(213) \quad c'' = -\frac{a_x}{a}c' - \frac{1}{2}\frac{a_c}{a}c'^2 - \frac{\lambda}{a}.$$

This proof uses only the weak Euler equation and the one-dimensional embedding $W^{1,1} \hookrightarrow C^0$; no separate truncation theorem is needed.

C.5. *Continuity in the phase.* Let c_Y be a minimizer at a lifted phase Y and put $c_{Y,h} = c_Y + h/\ell$. Then $\Pi_{m,\ell}(c_{Y,h}) = Y + h$. Since the derivative is unchanged,

$$E_{m,\ell}(Y + h) \leq \frac{1}{2\ell} \int_0^\ell a(x, c_Y + h/\ell) c_Y'^2 dx.$$

The mean-value theorem gives

$$E_{m,\ell}(Y + h) - E_{m,\ell}(Y) \leq C|h| \int_0^\ell c_Y'^2 dx.$$

An affine competitor bounds the minimum uniformly in Y , and $a \geq a_-$ bounds $\int c_Y'^2$. Reversing the roles of Y and $Y + h$ gives a local Lipschitz estimate.

C.6. *The multiplier and phase response.* Suppose on an interval of phases there is a differentiable family c_Y of minimizers, normalized so that

$$\frac{d}{dY} \Pi_{m,\ell}(c_Y) = 1.$$

Differentiate the minimized action. The interior first variation reduces to the multiplier term:

$$\begin{aligned} \frac{d}{dY} E_{m,\ell}(Y) &= \frac{1}{\ell} \int_0^\ell \left[-\frac{d}{dx}(a c_Y') + \frac{1}{2} a_c c_Y'^2 \right] \partial_Y c_Y dx \\ &= \frac{\lambda}{\ell} \int_0^\ell \partial_Y c_Y dx. \end{aligned}$$

From the phase normalization,

$$\int_0^\ell \partial_Y c_Y dx = 1$$

under the conventions of (96). Thus the derivative is a fixed multiple of λ . This confirms that flatness forces the multiplier to vanish; the comparison proof in Proposition 4.6 requires no branch differentiability and is therefore preferred in the main argument.

Appendix D. Crossing, Birkhoff order, and a priori compactness

This appendix supplies the one-dimensional order and compactness statements used in Sections 6 and 8. The point of writing them separately is that none of them uses differentiability with respect to the rotation number.

D.1. *Min-max splicing in one winding class.* Let $c_1, c_2 \in \mathcal{C}_{m,\ell}$ minimize the full winding action. Put

$$c^- := \min(c_1, c_2), \quad c^+ := \max(c_1, c_2).$$

Both functions have the same winding law. Almost everywhere, the unordered pairs

$$(c^-, (c^-)'), \quad (c^+, (c^+)')$$

and

$$(c_1, c_1'), \quad (c_2, c_2')$$

coincide. Therefore

$$(214) \quad \mathcal{A}_{m,\ell}(c^-) + \mathcal{A}_{m,\ell}(c^+) = \mathcal{A}_{m,\ell}(c_1) + \mathcal{A}_{m,\ell}(c_2).$$

Each term on the left is at least the full-class minimum, so equality of the sum forces both c^- and c^+ to be minimizers. They are smooth. If the original curves met transversely, one of the spliced curves would have a corner. Hence the derivatives agree at every meeting point, and uniqueness for the Euler ODE gives $c_1 \equiv c_2$. This proves Lemma 6.1.

D.2. *Birkhoff order and bounded deviation.* Every integer deck translate of a full-class minimizer is another full-class minimizer of the same rotation. More precisely, if

$$\rho = -\frac{m}{\ell}, \quad c(x + \ell) = c(x) + \rho\ell,$$

then

$$c_{j,k}(x) := c(x + j) + k, \quad (j, k) \in \mathbb{Z}^2,$$

has the same winding and the same action. The no-crossing argument therefore orders $c_{j,k}$ and c globally.

Write $c(x) = \rho x + w(x)$ with w ℓ -periodic. A change of variables shows

$$(215) \quad \frac{1}{\ell} \int_0^\ell (c(x + j) + k - c(x)) dx = \rho j + k.$$

If the two curves are distinct, their pointwise difference has one strict sign, so its sign is the sign of the mean in (215). If the mean is zero, the curves coincide. Choosing the two consecutive integers which bracket $-\rho j$ gives the standard Birkhoff estimate

$$(216) \quad |c(x + j) - c(x) - \rho j| \leq 1 \quad (x \in \mathbb{R}, j \in \mathbb{Z}).$$

In particular,

$$(217) \quad |c(x+1) - c(x)| \leq |\rho| + 1.$$

The same estimate passes to every locally uniform limit of these minimizers.

D.3. *A priori velocity bounds.*

LEMMA D.1 (Uniform Tonelli bound on bounded rotation sets). *For every $R < \infty$ there is $C_R < \infty$, depending only on R and on finitely many bounds for a and a^{-1} , such that every full-class rational minimizer with $|\rho| \leq R$ satisfies*

$$|c'(x)| + |c''(x)| \leq C_R \quad \text{for all } x \in \mathbb{R}.$$

The same bound holds for every local C^1 limit of these minimizers.

Proof. Fix an interval $[x_0, x_0 + 1]$ and put $D = c(x_0 + 1) - c(x_0)$. By (217), $|D| \leq R + 1$. The affine segment joining the endpoints is an admissible fixed-endpoint competitor. A full-class primitive-cell minimizer already minimizes every subarc of length at most ℓ : otherwise replacing that subarc, and leaving the rest of the primitive cell unchanged, would lower the full-cell action. Since $\ell \geq 1$, this applies to the present unit interval. Thus

$$\frac{a_-}{2} \int_{x_0}^{x_0+1} c'^2 dx \leq \int_{x_0}^{x_0+1} \frac{1}{2} a(x, c) c'^2 dx \leq \frac{a_+}{2} D^2.$$

Thus the L^2 norm of c' on every unit interval is bounded uniformly.

The Euler equation is

$$(218) \quad c'' = -\frac{a_x}{a} c' - \frac{1}{2} \frac{a_c}{a} c'^2.$$

Hence $y = c'$ satisfies

$$|y'| \leq C_0(|y| + y^2).$$

Suppose $|y(x_*)| = M \geq 2$. Comparison with the scalar equation $z' = 2C_0 z^2$ shows that, on an interval of length at least c_0/M on one side of x_* , one has $|y| \geq M/2$. That interval is contained in the union of at most two unit intervals. Its contribution to the already established local L^2 bound is at least $c_1 M$. Therefore M is uniformly bounded. Equation (218) then bounds c'' .

For a locally C^1 convergent sequence of these minimizers, the estimate passes to the limit. Repeated differentiation of the ODE gives higher local bounds whenever needed. $\square$

The same conclusion is often quoted as the Tonelli a priori compactness lemma. We included the unit-interval proof because it makes clear where one-dimensional Birkhoff order enters and, in particular, why the estimate is independent of the primitive denominator ℓ .

D.4. *A uniform modulus for the rational momentum graphs.*

LEMMA D.2 (Uniform rational graph modulus). *For the smooth invariant graphs furnished by flat rational cells, and for every $R < \infty$, there is $K_R < \infty$ such that*

$$\|P_q\|_\infty + \|\partial_x P_q\|_\infty + \|\partial_c P_q\|_\infty \leq K_R \quad (q \in \mathbb{Q}, |q| \leq R).$$

In particular the bound is independent of the primitive denominator.

Proof. Write the Euler equation as $c'' = f(x, c, c')$, where

$$f(x, c, v) = -\frac{a_x(x, c)}{a(x, c)}v - \frac{a_c(x, c)}{2a(x, c)}v^2.$$

Lemma D.1 bounds velocities uniformly for $|q| \leq R$. Therefore there are constants $A_R, B_R < \infty$ bounding $|f_c|$ and $|f_v|$ along all these leaves.

Let $U(x, t)$ be a smooth normalized rational family and $\eta(x) = U_t(x, t) > 0$. Its Jacobi equation gives, for $z = \eta'/\eta$,

$$z' = f_c + f_v z - z^2 \leq A_R + B_R |z| - z^2.$$

Choose $K > B_R + \sqrt{B_R^2 + 2A_R}$. For $|z| \geq K$, the right side is at most $-z^2/2$. A value $z > K$ would force a finite backward pole; a value $z < -K$ would force a finite forward pole. Both are impossible because η is a positive entire Jacobi field. Thus $|z| \leq K$, uniformly in the rotation and the leaf.

Writing $V_q(x, U(x, t)) = U_x(x, t)$ gives

$$\partial_c V_q(x, U(x, t)) = \frac{U_{xt}(x, t)}{U_t(x, t)} = z(x),$$

and the invariant-graph equation gives

$$\partial_x V_q = f(x, c, V_q) - V_q \partial_c V_q.$$

The velocity bound and the bound on z therefore control both first derivatives of V_q . Since $P_q = aV_q$ and a is fixed and smooth, the claimed momentum estimate follows. $\square$

D.5. *Where order between rotations enters.* The within-class min-max argument above is used for winding minimizers before they are known to be class-A minimizing. Order between distinct rational rotations is used only after Hamiltonian calibration has proved class-A minimality. Proposition 8.1 derives that order directly from the two-crossing principle and the different asymptotic slopes. No graph theorem for static or Aubry sets is needed.

Appendix E. Closed calibrations, weak Stokes, and rational uniqueness

This appendix supplies the differential-form and return-map details used in Sections 2, 3, 6, and 7. The main text keeps the square identities; here we justify the endpoint and homology statements behind them.

E.1. *The counterexample calibration as a closed one-form.* For the potential of Section 2, fix $E \geq 0$ and $\sigma \in \{-1, +1\}$ and set

$$(219) \quad \omega_{E,\sigma} := \sigma g_E(u) du - E dx.$$

Because g_E is smooth and 1-periodic, $\omega_{E,\sigma}$ is a smooth periodic one-form on $\mathbb{T}^2$, and

$$d\omega_{E,\sigma} = 0.$$

On the graph of an absolutely continuous function $u = w(x)$,

$$\omega_{E,\sigma}|_{\text{graph } w} = (\sigma g_E(w)w' - E) dx.$$

Therefore the algebraic identity (27) is exactly

$$(220) \quad \left(\frac{1}{2}w'^2 - V_*(w) \right) dx = \omega_{E,\sigma}|_{\text{graph } w} + \frac{1}{2}(w' - \sigma g_E(w))^2 dx.$$

On the universal cover, the primitive of the du term is $\sigma G_E(u)$, so fixed endpoints give the same one-form integral. On a closed quotient, homologous curves give the same integral by Stokes. This is the differential-form version of the endpoint calculation in (30); no exactness of the form on the torus is asserted or needed.

E.2. *Stokes for Sobolev graph strips.* Let

$$\omega = A(x, u) dx + C(x, u) du$$

be a smooth closed one-form on a simply connected strip. If two $W^{1,2}$ graphs u_0, u_1 have the same endpoints, approximate them in $W^{1,2}$ by smooth graphs with the same endpoints. For smooth graphs, their difference is the oriented boundary of a Lipschitz two-chain, and Stokes' theorem gives

$$\int_{\text{graph } u_1} \omega - \int_{\text{graph } u_0} \omega = \int_{\Sigma} d\omega = 0.$$

Line integration is continuous under $W^{1,2}$ convergence because the coefficients are bounded and Lipschitz on compact sets. Passing to the limit proves the same statement for the Sobolev graphs.

On a rational quotient, two closed graphs in the same homology class differ by the boundary of a two-chain. The identical argument gives equality of their periods. This is the precise form of Stokes used in Proposition 3.7 and Proposition 3.8.

E.3. *Closed Hamiltonian graph forms.* Let P be a locally Lipschitz invariant momentum graph and assume

$$P_x + H_p(x, c, P)P_c + H_c(x, c, P) = 0$$

almost everywhere. Define

$$\lambda = P dc - H(x, c, P) dx.$$

Distributional differentiation gives

$$d\lambda = (P_x + H_p P_c + H_c) dx \wedge dc = 0.$$

A closed L_{loc}^∞ one-form on the simply connected plane has a locally Lipschitz primitive. One direct construction is to mollify λ , integrate the mollifications along axis-parallel polygonal paths, and use the vanishing circulation around rectangles to pass to a locally uniform limit. The weak differential of the limit is λ .

In the flat-cell construction the graph is in fact smooth by Proposition 6.5, so the classical Poincaré lemma already suffices. The weak version is retained because one-sided limits in the rotation parameter are only Lipschitz.

E.4. *Transverse regularity of a flat-cell foliation.* Let $U(x, t)$ be the flat-cell family relabelled by $U(0, t) = t$. Then

$$U(x + \ell, t) = U(x, t) - m, \quad U(x, t + 1) = U(x, t) + 1.$$

The family is continuous in C^1 on compact x -intervals by Proposition 6.4. Write its Euler equation as

$$U_{xx} = f(x, U, U_x), \quad f(x, c, v) = -\frac{a_x}{a}v - \frac{a_c}{2a}v^2.$$

For $s \neq 0$, set

$$\phi_s(x, t) = \frac{U(x, t + s) - U(x, t)}{s} > 0, \quad \phi_s(0, t) = 1.$$

These functions are ℓ -periodic in x . The mean-value formula gives

$$\phi_s'' = b_s \phi_s' + d_s \phi_s,$$

where, with $U_\theta = (1 - \theta)U(\cdot, t) + \theta U(\cdot, t + s)$,

$$b_s(x, t) = \int_0^1 f_v(x, U_\theta, U_\theta') d\theta, \quad d_s(x, t) = \int_0^1 f_c(x, U_\theta, U_\theta') d\theta.$$

Lemma D.1 makes these coefficients uniformly bounded for small s , independently of x . The logarithmic derivative $z_s = \phi_s'/\phi_s$ satisfies

$$z_s' = d_s + b_s z_s - z_s^2.$$

If $|d_s| \leq A$ and $|b_s| \leq B$, choose $K > B + \sqrt{B^2 + 2A}$. Outside $[-K, K]$ the right side is at most $-z_s^2/2$. Positivity on the entire line excludes the resulting backward or forward pole, so $|z_s| \leq K$. Normalization gives $e^{-K\ell} \leq \phi_s \leq e^{K\ell}$

on $[0, \ell]$, and the equation bounds its first and second derivatives. The difference quotients are therefore precompact in $C^1([0, \ell])$.

Every limit is a positive periodic Jacobi field along $U(\cdot, t)$, normalized to equal one at $x = 0$. The Jacobi equation is

$$J\phi = -(A_0\phi')' + W_0\phi = 0, \quad A_0 = a(x, U),$$

$$W_0 = \frac{1}{2}a_{cc}(x, U)U_x^2 - \frac{d}{dx}(a_c(x, U)U_x).$$

If ϕ_1, ϕ_2 are two positive periodic solutions, their weighted Wronskian $A_0(\phi_1'\phi_2 - \phi_1\phi_2')$ is constant. A nonzero value would make the periodic ratio ϕ_1/ϕ_2 strictly monotone. Thus they are proportional, and normalization makes them equal. It follows that all difference quotients converge to the same field $U_t > 0$. Applying the same compactness argument when $(t_j, s_j) \rightarrow (t, 0)$ gives joint continuity of U_t and U_{xt} .

Now apply Lemma B.3 to L_a on $[0, \ell]$, using this positive Jacobi field. The prescribed endpoint values are $U(0, t) = t$ and $U(\ell, t) = t - m$. The lemma proves $U \in C^\infty(\mathbb{R}^2)$, with extension to the full plane supplied by the winding law. This proves Proposition 6.5 without an assumed transverse differentiability or an unproved higher-difference-quotient estimate.

E.5. *The calibrated quadratic gap.* For $L_a = \frac{1}{2}av^2$, $H = p^2/(2a)$, and a graph momentum $P = aV$, one has

$$\begin{aligned} (221) \quad L_a(x, c, v) - (Pv - H(x, c, P)) &= \frac{1}{2a}(av - P)^2 \\ &= \frac{1}{2}a(x, c)(v - V(x, c))^2. \end{aligned}$$

Let $U(x, t)$ be a smooth parameterization of the calibrated rational foliation and let $T(x, c)$ be its inverse phase. Along a reference leaf,

$$T_x + VT_c = 0.$$

For an arbitrary curve $c(x)$ put $t(x) = T(x, c(x))$. Then

$$t' = T_x + T_c c' = T_c(c' - V).$$

Substitution into (221) gives

$$(222) \quad L_a - \lambda|_{\text{graph } c} = \frac{1}{2} \frac{a(x, c)}{T_c(x, c)^2} t'(x)^2.$$

The coefficient is smooth and bounded above and below by positive constants on the compact rational quotient.

E.6. *The corrected rational return map.* Fix the reference rotation $\rho = -m/\ell$. Let $c(x)$ be an orbit of a second full invariant minimizing graph of the same rotation, and define

$$t(x) := T(x, c(x)).$$

At $x = 0$ the normalization gives $T(0, c) = c$. The reference deck law implies

$$T(x + \ell, c) = T(x, c + m) = T(x, c) + m.$$

Consequently the map taking $t(0)$ to $t(\ell)$ is exactly the lift obtained from the competing time- ℓ return map after adding the vertical integer m . It is an increasing degree-one circle homeomorphism and its translation number is

$$\ell\rho + m = 0.$$

An increasing degree-one lift has translation number zero if and only if it has a fixed point. If s_* is one fixed point, then the consecutive fixed points $s_* + j$ bracket every orbit. After adding one fixed integer to the initial lift if necessary, we have

$$(223) \quad \sup_{n \in \mathbb{Z}} |t(n\ell) - t(0)| \leq 1.$$

This is the bounded-endpoint statement used in the main proof.

E.7. *Affine comparison and rational uniqueness.* On $[-N\ell, N\ell]$, let $\bar{t}_N$ be the affine function joining $t(-N\ell)$ and $t(N\ell)$, and set

$$\bar{c}_N(x) := U(x, \bar{t}_N(x)).$$

The physical endpoints of $\bar{c}_N$ agree with those of c . Since the competing orbit is globally minimizing, subtraction of the closed reference calibration and (222) give

$$(224) \quad \int_{-N\ell}^{N\ell} \frac{1}{2} \frac{a(x, c)}{T_c(x, c)^2} t'^2 dx \leq \int_{-N\ell}^{N\ell} \frac{1}{2} \frac{a(x, \bar{c}_N)}{T_c(x, \bar{c}_N)^2} \bar{t}_N'^2 dx.$$

By (223), the endpoint difference is bounded independently of N . If M is an upper bound for the coefficient, then

$$\text{right side of (224)} \leq \frac{M}{4N\ell} (t(N\ell) - t(-N\ell))^2 \longrightarrow 0.$$

The left side is nonnegative. Its integral on every fixed compact interval therefore vanishes, so $t' = 0$ almost everywhere and hence everywhere. The competing orbit is a reference leaf. Applying this to every orbit of the competing full graph proves Proposition 7.4.

Appendix F. Translation numbers and elementary Hamiltonian estimates

This appendix supplies the circle-map facts, completeness, area preservation, and properness used in Section 8. It makes no twist-map assertion about a time-one flow map.

F.1. *Increasing lifts and translation number.*

LEMMA F.1 (Poincare translation number). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing homeomorphism with $f(s+1) = f(s) + 1$. Then*

$$\rho(f) := \lim_{n \rightarrow \infty} \frac{f^n(s) - s}{n}$$

exists, is independent of s , and has the following properties.

- (i) $|f^n(s) - s - n\rho(f)| \leq 1$ for every $n \in \mathbb{Z}$.
- (ii) Uniform convergence $f_j \rightarrow f$ implies $\rho(f_j) \rightarrow \rho(f)$.
- (iii) $\rho(f) = 0$ if and only if f has a fixed point. If $f^q(s) = s + p$, with $q \geq 1$ and $p \in \mathbb{Z}$, then $\rho(f) = p/q$.
- (iv) Translation number is unchanged by conjugacy by an increasing degree-one lift, and $\rho(f^q - p) = q\rho(f) - p$.
- (v) If $f < g$ pointwise and $\rho(f)$ is irrational, then $\rho(f) < \rho(g)$.

Proof. For $n \geq 1$, put

$$M_n = \max_s (f^n(s) - s), \quad m_n = \min_s (f^n(s) - s).$$

The functions are periodic in s . Monotonicity and degree one give $M_n - m_n \leq 1$. Moreover $M_{n+k} \leq M_n + M_k$ and $m_{n+k} \geq m_n + m_k$. The subadditive and superadditive limit arguments therefore give one common limit ρ , with $m_n \leq n\rho \leq M_n$. This proves existence and the bound in (i) for $n > 0$. Applying that bound at $f^{-n}(s)$ proves it for negative n .

For (ii), first fix n large enough that $2/n$ is small. Uniform convergence of the lifts gives uniform convergence of that fixed iterate, and (i) bounds the error between each translation number and its n -step displacement divided by n . A fixed point forces zero translation number. If there is no fixed point, the continuous periodic function $f(s) - s$ has one strict sign and is bounded away from zero; iterating proves a translation number of that sign. Iterating $f^q(s) = s + p$ proves the last assertion in (iii). For (iv), a degree-one conjugating lift differs from the identity by a bounded function. Conjugate orbit displacements therefore differ by a bounded amount. The iterate-and-integer formula follows from $(f^q - p)^n = f^{qn} - np$.

For (v), set $\alpha = \rho(f)$ and $\delta = \min_s (g(s) - f(s)) > 0$. Induction gives $g^n(s) \geq f^n(s) + \delta$ for every $n \geq 1$. Since α is irrational, choose integers $q \geq 1, p$

with $0 < p - q\alpha < \delta$. Since $M_q \geq q\alpha$, there is an s_0 such that

$$g^q(s_0) \geq f^q(s_0) + \delta \geq s_0 + q\alpha + \delta > s_0 + p.$$

The lift $h = g^q - p$ thus satisfies $h(s_0) > s_0$. Its forward iterates at s_0 are increasing, so $\rho(h) \geq 0$. By (iv), $q\rho(g) - p \geq 0$, and hence $\rho(g) \geq p/q > \alpha$. $\square$

F.2. Completeness and area preservation.

LEMMA F.2 (Complete area-preserving section dynamics). *For every smooth positive periodic a , the Hamiltonian $H(x, c, p) = p^2/(2a(x, c))$ generates a complete flow. Every integer-time section map is an area-preserving diffeomorphism of the annulus $\mathbb{T}_c \times \mathbb{R}_p$ which preserves both ends.*

Proof. Along an orbit, the total derivative of the Hamiltonian is its explicit time derivative:

$$(225) \quad \frac{d}{dx} H(x, c(x), p(x)) = H_x(x, c(x), p(x)) = -\frac{a_x(x, c(x))}{a(x, c(x))} H.$$

For $M = \|a_x/a\|_\infty$, this gives

$$e^{-M|t|} H(x_0) \leq H(x_0 + t) \leq e^{M|t|} H(x_0).$$

Since a is bounded above and below, momentum remains bounded on every finite time interval. Thus no finite-time escape is possible and the flow is complete. The set $p = 0$ is invariant; uniqueness shows that any other orbit retains the sign of its momentum.

The vector field $(H_p, -H_c)$ has zero divergence in (c, p) , so its section maps preserve $dc dp$. The preceding estimates also give

$$\sqrt{\frac{a_-}{a_+}} e^{-M|t|/2} |p(x_0)| \leq |p(x_0 + t)| \leq \sqrt{\frac{a_+}{a_-}} e^{M|t|/2} |p(x_0)|.$$

Together with sign preservation, this proves preservation of each annulus end. Smooth dependence and the backward flow give the diffeomorphism property. $\square$

F.3. Quantitative properness of graph rotation.

LEMMA F.3 (Properness without a normal-form argument). *Let $p = P(x, c)$ be a full invariant Lipschitz graph with rotation ρ . Put $M = \|a_x/a\|_\infty$ and $c_* = a_- e^{-M/2}$. Uniformly in (x, c) ,*

$$P(x, c) \geq c_*(\rho - 1) \quad (\rho > 1), \quad P(x, c) \leq -c_*(|\rho| - 1) \quad (\rho < -1).$$

Proof. Choose an orbit on the graph through (x_0, c_0, p_0) , and set $D = c(x_0 + 1) - c_0$. On the graph, return lifts at different time sections are conjugate by the intervening graph flow, which is an increasing degree-one lift. They have the same translation number ρ . Lemma F.1 therefore gives $|D - \rho| \leq 1$.

For $0 \leq t \leq 1$, (225) yields

$$|c'(x_0 + t)| = \sqrt{\frac{2H(x_0 + t)}{a(x_0 + t, c(x_0 + t))}} \leq \frac{e^{M/2}}{a_-} |p_0|.$$

Consequently $|D| \leq e^{M/2} |p_0| / a_-$. Momentum retains its sign, and $c' = p/a$, so a nonzero displacement has the same sign as p_0 . If $\rho > 1$, then $D \geq \rho - 1 > 0$, giving $p_0 \geq a_- e^{-M/2} (\rho - 1)$. The case $\rho < -1$ is identical with the opposite sign. $\square$

Appendix G. Index forms, Jacobi fields, and strict domain monotonicity

We prove the spectral step in Theorem 9.3 and record the invariance of conjugate points under the drift normal form.

G.1. *The Jacobi operator for L_a .* For $L_a(x, c, v) = \frac{1}{2}a(x, c)v^2$, the Hessian entries are

$$L_{vv} = a, \quad L_{cv} = a_c v, \quad L_{cc} = \frac{1}{2} a_{cc} v^2.$$

Along an extremal c , the second variation is

$$I[\eta] = \int \left(a\eta'^2 + 2a_c c' \eta \eta' + \frac{1}{2} a_{cc} c'^2 \eta^2 \right) dx.$$

Integrating the mixed term by parts for zero endpoint variations gives

$$(226) \quad I[\eta] = \int \left(a\eta'^2 + \left[\frac{1}{2} a_{cc} c'^2 - (a_c c')' \right] \eta^2 \right) dx.$$

Thus the self-adjoint Jacobi operator is

$$(227) \quad \mathcal{J}_c \eta = -(a\eta')' + \left[\frac{1}{2} a_{cc} c'^2 - (a_c c')' \right] \eta.$$

The linearized Euler equation is $\mathcal{J}_c \eta = 0$.

G.2. *Conjugacy and the first eigenvalue.*

LEMMA G.1 (Conjugate endpoints give a nonpositive eigenvalue). *If a nonzero Jacobi field vanishes at $r < s$, then the first Dirichlet eigenvalue of $\mathcal{J}_c$ on $[r, s]$ is at most zero.*

Proof. The Jacobi field is an admissible test function and integration by parts gives $I[\eta] = 0$. The Rayleigh quotient therefore has infimum at most zero. $\square$

LEMMA G.2 (Strict Dirichlet domain monotonicity). *If $[r, s]$ is properly contained in $[r', s']$, then*

$$\lambda_1(r', s') < \lambda_1(r, s)$$

for every regular scalar Sturm–Liouville operator with positive leading coefficient.

Proof. Extend the positive first eigenfunction on $[r, s]$ by zero. This gives the weak inequality for the Rayleigh infima. If equality held, the zero extension would be a first eigenfunction on the larger interval. A first eigenfunction is strictly positive in the interior and C^1 , whereas the zero extension vanishes on a nonempty open interval and has an interior corner unless its normal derivative vanishes. Unique continuation for the scalar ODE excludes either possibility. $\square$

If a globally minimizing extremal had conjugate endpoints, nonnegative second variation would make the first eigenvalue zero there, while strict domain monotonicity would make it negative on a larger interval. This contradicts minimality on the larger interval.

G.3. Point transformations and Jacobi fields. Let $(x, c) \mapsto (y, u)$ be the normal-form diffeomorphism and let c_ε be a smooth family of extremals. Differentiating $u_\varepsilon(Y(x)) = F(x, c_\varepsilon(x))$ gives

$$\xi_u(y) = F_c(x, c(x))\xi_c(x).$$

Since $F_c > 0$, the map between Jacobi fields is pointwise invertible and preserves their zeros. The positive base reparametrization $y = Y(x)$ also preserves endpoint order. Hence conjugate points and their multiplicities are invariant under the normal form.

Appendix H. Green solutions and weak Hopf integration

We supply the analytic details behind Proposition 11.3 and Proposition 11.4.

H.1. Finite-horizon Jacobi fields. Let $k(t)$ be a bounded continuous coefficient along a complete orbit, and assume the equation $J'' = kJ$ has no conjugate points. For $T > 0$, the solution $J_T(-T) = 0$, $J_T(0) = 1$ is positive on $(-T, 0]$. Its logarithmic derivative $w_T = J'_T/J_T$ solves

$$w'_T + w_T^2 - k = 0.$$

If $K_0 = \|k\|_\infty$, comparison with the constant-coefficient equations gives

$$(228) \quad -\sqrt{K_0} \leq w_T(0) \leq \begin{cases} \sqrt{K_0} \coth(\sqrt{K_0}), & K_0 > 0, \\ 1, & K_0 = 0, \end{cases} \quad T \geq 1.$$

For the upper bound, compare backward with the Riccati solution having a pole at time $-T$; for the lower bound, a value below $-\sqrt{K_0}$ would force a future pole and hence a future zero of J_T .

The Wronskian calculation in Lemma 11.1 proves monotonicity in T . Thus the Green slope exists at every phase point.

H.2. *Consistency under the flow.* Let $z_s = \Phi^s z$. The finite-horizon Jacobi problem based at z_s is the time translate of the problem based at z , with past horizon shifted by s . Passing to the monotone limit shows that the global positive Green Jacobi fields are compatible under time translation. Hence their logarithmic derivative defines one function r on phase space, not merely a family of orbitwise functions.

On a compact time interval, normalize the finite-horizon Jacobi fields at one point. The Riccati bounds give uniform C^1 control, and the equation gives C^2 control. The monotone slope limit identifies every subsequential limit. The limiting positive field satisfies the Jacobi equation, so its logarithmic derivative is C^1 along the orbit and solves the Riccati equation.

H.3. *Borel measurability and the distributional equation.* For finite horizon T , the Jacobi solution depends continuously on the initial phase point as long as the past interval is fixed. Thus r_T is continuous. The monotone limit $r = \inf_{T \geq 1} r_T$ is Borel.

Let $\mathcal{U}$ be a flow box with coordinates (s, ζ) such that $D = \partial_s$. The orbitwise identity

$$\partial_s r + r^2 - K = 0$$

holds for almost every transverse parameter ζ and every s . Multiplication by a smooth compactly supported test function and Fubini's theorem gives the weak identity in the flow box. A partition of unity gives it globally.

H.4. *Cutoff proof of the slab identity.* Choose a smooth cutoff $\chi_{R,\varepsilon}(v)$ which equals one on $[-R, R]$, vanishes outside $[-R - \varepsilon, R + \varepsilon]$, and converges to the characteristic function of $[-R, R]$. Test

$$Dr = K - r^2$$

against $\chi_{R,\varepsilon}$. Since D is divergence-free,

$$\begin{aligned} \int \chi_{R,\varepsilon} r^2 &= \int \chi_{R,\varepsilon} K + \int r D \chi_{R,\varepsilon} \\ &= \int \chi_{R,\varepsilon} K + \int r Q_B \chi'_{R,\varepsilon}(v). \end{aligned}$$

The first term tends to zero because $\int_{\mathbb{T}^2} K = \int_{\mathbb{T}^2} (Q_B)_u = 0$ at every velocity. For almost every R , the coarea/Lebesgue differentiation theorem sends the second term to

$$\int_{\mathbb{T}^2} Q_B r(\cdot, -R) - \int_{\mathbb{T}^2} Q_B r(\cdot, R).$$

The left side converges by dominated convergence because r is bounded. This proves the slab identity at Lebesgue radii.

H.5. *Absolute continuity of the slab energy.* Since $r^2 \in L^\infty$, Fubini gives

$$A(R) = \int_{-R}^R f(v) dv, \quad f(v) = \int_{\mathbb{T}^2} r(y, u, v)^2 dy du.$$

Thus A is locally absolutely continuous and $A'(R) = f(R) + f(-R)$ for almost every R . This justifies every differential step in the Hopf argument.

H.6. *The blow-up contradiction.* If a nondecreasing absolutely continuous function A satisfies $A^2 \leq CA'$ almost everywhere and $A(R_0) > 0$, then

$$(A^{-1})' = -A'/A^2 \leq -C^{-1}$$

almost everywhere on $[R_0, \infty)$. Integration contradicts positivity of A^{-1} at a finite radius. Hence $A \equiv 0$. No unproved decay of r at large velocity is required.

Appendix I. Normal-form computations and preservation laws

We collect the coordinate calculations behind Theorem 10.1 and the final separation step.

I.1. *Equivariance of the coordinate map.* From periodicity of a ,

$$I(x+1) = I(x).$$

The definition of F gives

$$F(x+1, c) = F(x, c), \quad F(x, c+1) = F(x, c) + 1.$$

The base map satisfies $Y(x+1) = Y(x) + 1$. Hence

$$(x, c) \mapsto (Y(x), F(x, c))$$

commutes with both deck transformations and descends to a torus diffeomorphism.

If $c(x+\ell) = c(x) - m$, then $y(x+\ell) = y(x) + \ell$ and

$$\begin{aligned} u(y+\ell) &= F(x+\ell, c(x+\ell)) \\ &= F(x, c(x) - m) = u(y) - m. \end{aligned}$$

Thus every rational rotation class is preserved exactly.

I.2. *Sobolev chain rule.* The configuration map and its inverse have bounded derivatives on the torus. They therefore map $W^{1,2}$ graphs to $W^{1,2}$ graphs and preserve endpoint traces. The chain rule holds almost everywhere. Formula (159) and the action identity extend from smooth curves by density.

I.3. *First variations.* Let c_ε be a fixed-endpoint variation and let u_ε be its transformed variation. Differentiating (160) gives

$$D\mathcal{A}_a[c](\eta_c) = S^{-1}D\mathcal{A}_B[u](\eta_u).$$

The linear map $\eta_c \mapsto \eta_u = F_c\eta_c$ is invertible and preserves zero endpoint values. Thus one first variation vanishes for all variations if and only if the other does.

Differentiating twice along an extremal gives the corresponding index-form identity

$$(229) \quad I_a[\eta_c] = S^{-1}I_B[F_c\eta_c].$$

This proves preservation of nonnegativity and of Jacobi kernels. In particular conjugate points are preserved.

I.4. *Euler equation for the drift Lagrangian.* For $L_B = \frac{1}{2}(v - B)^2$,

$$L_v = v - B, \quad L_u = -(v - B)B_u.$$

The Euler equation $dL_v/dy - L_u = 0$ is

$$v_y - B_y - B_u v + (v - B)B_u = 0,$$

which simplifies to

$$v_y = B_y + BB_u.$$

No term depending on v remains.

I.5. *Jacobi equation.* The Euler equation is $u_{yy} = Q_B(y, u)$. A variation $u + \varepsilon\xi$ gives

$$\xi_{yy} = (Q_B)_u \xi.$$

This agrees with the second variation of L_B after integration by parts.

I.6. *Derivative of the drift.* At fixed (y, u) , $x = X(y)$ and $c = C(x, u)$. Since $C_u = 1/F_c$,

$$B_u = SI(x)^2 F_{xc} C_u = SI(x)^2 \frac{F_{xc}}{F_c}.$$

Thus $B_u = 0$ if and only if $F_{xc} = 0$.

If $F_{xc} = 0$, then $F_c = G(c)$. The identity $F_c = \sqrt{a}/I$ yields $a = I^2 G^2$. Conversely, for a separated coefficient the vertical normalization makes F_c independent of x , so $B_u = 0$. Hence the normal form identifies separability exactly, not merely in one direction.

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