

Minimizing foliations: counterexamples and common-volume rigidity

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Abstract

We study minimizing graph foliations for the periodic action $\int(\frac{1}{2}|\nabla u|^2 - V(x, u)) dx$ in arbitrary base dimension $n \geq 2$. We first construct nonconstant potentials with ordinary smooth minimizing foliations at every slope. The failure of closed rational holonomy can be placed at any prescribed rational vector. We then prove an exact common-volume rigidity theorem: a prescribed set of at least two slopes forces $V_u = 0$ under one common positive invariant volume if and only if it is not contained in a Euclidean sphere with rational centre. Every excluded configuration is realized within the ordinary all-slope class. Phase locking and a mean-removing shear yield the criterion, and a spectral–Poisson construction describes all compatible families. We obtain optimal finite tests and an intrinsic obstruction at the rational centre of a full-rank family. The appendices establish rational-cell and quantitative rigidity results and extend the compatibility calculation to uniformly convex kinetics and nonlinear elliptic selection. The global implication from rational identity holonomy alone is not asserted.

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1. Introduction

We consider

$$(1.1) \quad \Delta u = -V_u(x, u), \quad V \in C^\infty(\mathbb{T}^n \times \mathbb{T}), \quad n \geq 2,$$

with the fixed Euclidean action

$$\mathcal{A}_V[u; \Omega] = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - V(x, u) \right) dx.$$

The question is whether minimizing graph foliations at every slope force $V_u = 0$. It appears in [4, Section 5] and as Question 4 in [1].

First we have to account for the fact that a rational foliation can have nontrivial holonomy on its leaf space. In an ordinary foliation, deck translations preserve the family, but they need not preserve each rational leaf with its prescribed vertical increment. These are different conditions. We give an explicit counterexample under the ordinary convention, and then prove an exact rigidity theorem under one common invariant-volume condition.

1.1. *The counterexample.* Let $b \geq 0$ be smooth and one-periodic, nonzero but with at least one zero. The potential

$$V_b(u) = -\frac{1}{2}b(u)^2$$

has a smooth minimizing foliation at every slope. At a nonzero slope we use rotating profiles; at slope zero we use the bounded flow of $y' = b(y)$. The same square completion proves minimality against competitors depending on all n variables.

For $b(u) = \sin^2(\pi u)$ the zero-slope return map is not the identity. More importantly, there is no alternative closed minimizing foliation at that slope: every periodic class-A minimizer is an integer constant. A rational shear moves this obstruction to any prescribed $\beta = -k/\ell \in \mathbb{Q}^n$ and gives

$$V_{k,\ell}(x, u) = -\frac{1}{2\ell^2} \sin^4(\pi(\ell u + k \cdot x)).$$

All slopes other than β still admit rigid minimizing foliations. Thus testing rigid holonomy at a prescribed set which omits even one rational vector cannot force rigidity. The precise convention and all comparison arguments are given in Sections 2 and 3.

1.2. *The common-volume theorem.* For a selected foliation of slope a , write $p_a(x, u)$ for the gradient of the leaf through (x, u) . The additional hypothesis is that one positive probability density $q \in C^1(\mathbb{T}^n \times \mathbb{T})$ satisfies

$$(1.2) \quad \partial_{x_i} q + \partial_u (qp_{a,i}) = 0 \quad (a \in S, 1 \leq i \leq n).$$

The density is not assumed constant. Nor do we identify densities chosen independently for different foliations.

For a set S containing at least two distinct slopes, put

$$\mathcal{C}_{\mathbb{Q}}(S) = \{\beta \in \mathbb{Q}^n : |a - \beta|^2 = |b - \beta|^2 \text{ for all } a, b \in S\}.$$

Theorem 6.1 proves that the universal rigidity statement under (1.2) holds exactly when

$$\mathcal{C}_{\mathbb{Q}}(S) = \emptyset.$$

In other words, the only slope configurations which permit a nontrivial common-volume family are those contained in a Euclidean sphere with rational centre. Every excluded configuration is realized by a potential which still has ordinary minimizing foliations at every real slope.

The proof first turns invariance into closed phase forms,

$$d\tau_a = q(du - p_a \cdot dx), \quad \tau_a = u - a \cdot x + \psi_a(x, u).$$

Comparing two residual equations shows that $\psi_b - \psi_a$ is constant. After a mean-removing shear, their common correction χ satisfies

$$(b - a) \cdot \nabla_x \chi + \frac{|b|^2 - |a|^2}{2} \chi_s = 0.$$

A single nonzero Fourier coefficient gives a rational centre for the whole collection. This proves the rigidity direction without differentiating the foliations in slope. The converse is the calibrated rational-shear construction above.

Two suitably chosen real slopes suffice; no single slope does. For rational slopes the smallest test has three points, and any three distinct collinear points work. Every sufficient set contains a sufficient subset of at most $n+2$ slopes. This bound is attained by inclusion-minimal rational tests. These are exact optimality statements within the common-volume class, not a claim that common volume is the weakest possible geometric hypothesis.

Section 7 reconstructs every common-volume family, including those whose slopes do not affinely span $\mathbb{R}^n$. Its only horizontal correction is the solution

of a Poisson equation. In full affine rank, the nontrivial family is a rational travelling wave,

$$q(x, u) = r(\ell u + k \cdot x), \quad V(x, u) = A(x) - \frac{R^2}{2q(x, u)^2},$$

and the rational centre cannot itself carry a closed minimizing foliation.

1.3. *Rational identity holonomy and the extensions.* Write $\text{RH}_{\mathbb{Q}}$ for the existence of a closed minimizing foliation at every rational slope, with identity adjusted holonomy on its natural finite cover. The local theorem in Appendix C gives

$$\text{RH}_{\mathbb{Q}} \quad \text{and} \quad \|V_u\|_{C^{2n+6}} < \delta_n \quad \implies \quad V_u = 0.$$

The unrestricted implication $\text{RH}_{\mathbb{Q}} \implies V_u = 0$ is not established here. In particular, we do not infer a common density or smooth dependence in slope from separately existing rational foliations.

The independent Euclidean results are placed in Appendices A–C: the rational cell criterion, exact pinning, quantitative phase escape, and local rigidity. The extensions to constant anisotropic forms, uniformly convex kinetics, elliptic selection, analytic compatibility and kinetic jets are in Appendices D–H. The variable-coefficient examples and their coordinate normal form are in Appendix I. The remaining appendices give the comparison, response, symmetry, and differential identities used in those arguments.

1.4. *Relation to earlier work.* The ordered variational setting and Hilbert calibration are classical; see [5], [2], and [4]. Our main calculation is the common-volume compatibility reduction and the reconstruction of its exceptional families. The rational gap and pinning arguments are related to [3] and [9]; the fixed-cell coupling analysis should also be compared with [8]. Analytic dependence of elliptic correctors has antecedents such as [7].

Nguyen’s min–max gap theorem [12] concerns minimal hypersurfaces. The non-rigidity results in [13] and [14] concern metric area and codimension-one stable norms. Those are different functionals from the fixed Euclidean quadratic action here. Likewise, no identification with cotangent-bundle Tonelli integrability [11] is used.

2. The full-dimensional counterexample

Let $n \geq 2$, let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and put

$$M = \mathbb{T}_x^n \times \mathbb{T}_u, \quad dm = dx \, du, \quad m(M) = 1.$$

For $V \in C^\infty(M)$, consider

$$(2.1) \quad \mathcal{A}_V[v; \Omega] = \int_{\Omega} \left(\frac{1}{2} |\nabla v|^2 - V(x, v) \right) dx, \quad \Delta v = Q(x, v), \quad Q = -V_u.$$

In particular,

$$(2.2) \quad \int_0^1 Q(x, u) du = 0 \quad \text{for every } x.$$

Definition 2.1 (Class-A minimizer). A function $v \in W_{\text{loc}}^{1,2}(\mathbb{R}^n)$ is class-A minimizing if, for every bounded Lipschitz $\Omega \Subset \mathbb{R}^n$,

$$\mathcal{A}_V[v; \Omega] \leq \mathcal{A}_V[v + \eta; \Omega] \quad \text{for all } \eta \in W_0^{1,2}(\Omega).$$

Definition 2.2 (Ordinary graph foliation). An ordinary smooth graph foliation of slope $\alpha \in \mathbb{R}^n$ is a smooth map $U : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ such that $U_t > 0$,

$$U(x, t + 1) = U(x, t) + 1, \quad \sup_x |U(x, t) - \alpha \cdot x| < \infty \quad \text{for every } t,$$

and, for every $m \in \mathbb{Z}^n$, there is an increasing degree-one smooth lift F_m with

$$(2.3) \quad U(x + m, t) = U(x, F_m(t)).$$

It is a solution foliation if every leaf solves (2.1), and a minimizing foliation if every leaf is class-A minimizing.

The family is preserved by deck transformations; individual rational leaves are not required to close. This is the ordered deck-invariant convention considered in the periodic variational theory of [5] and [2]. Fixed-trace and compactly supported comparison are equivalent; their approximation details are in Appendix J.

THEOREM 2.3 (All-dimensional counterexamples). *Let $b \in C^\infty(\mathbb{T})$ satisfy $b \geq 0$, $b \not\equiv 0$, and $b(s_*) = 0$ for some s_* . Then*

$$V_b(u) = -\frac{1}{2}b(u)^2$$

is nonconstant and has an ordinary smooth minimizing foliation at every slope. Every nonzero-slope foliation can be chosen with rigid deck law

$$(2.4) \quad U_\alpha(x + m, t) = U_\alpha(x, t + \alpha \cdot m).$$

There is no closed zero-slope minimizing foliation on a full-rank finite cover. For every prescribed $\alpha_0 = -k/\ell \in \mathbb{Q}^n$, the potential

$$(2.5) \quad V_{k,\ell}(x, u) = -\frac{1}{2\ell^2} \sin^4(\pi(\ell u + k \cdot x))$$

has ordinary minimizing foliations at every slope, rigid minimizing foliations at every $\alpha \neq \alpha_0$, and no closed minimizing foliation at α_0 .

Sections 2–3 prove this first. The calibration controls competitors depending on all n variables. The conclusion is negative under Definition 2.2; it does not refute a version of the question which already requires rational leaves to close.

Fix b as in Theorem 2.3. The equation is

$$(2.6) \quad \Delta v = b(v)b'(v),$$

and the action has nonnegative density

$$\mathcal{A}_b[v; \Omega] = \int_{\Omega} \left(\frac{1}{2} |\nabla v|^2 + \frac{1}{2} b(v)^2 \right) dx.$$

The construction uses one-dimensional profiles, but the minimality assertion is against arbitrary n -dimensional competitors. The calibration mechanism is classical; the issue is what it permits simultaneously at all slopes and at rational return maps.

2.1. *A square completion valid for every competitor.* For $E \geq 0$ put

$$g_E(s) = \sqrt{b(s)^2 + 2E}.$$

At $E = 0$ we take $g_0 = b$, which is smooth because $b \geq 0$. If $e \in \mathbb{R}^n$ is a unit vector, then for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$(2.7) \quad \frac{1}{2} |\xi|^2 + \frac{1}{2} b(s)^2 = \frac{1}{2} |\xi - eg_E(s)|^2 + eg_E(s) \cdot \xi - E.$$

Let $G_E(s) = \int_0^s g_E(a) da$. Since g_E and its derivative are bounded, G_E is globally Lipschitz and the Sobolev chain rule applies.

LEMMA 2.4 (Exact calibration against all-dimensional variations). *If v is a smooth entire solution of*

$$(2.8) \quad \nabla v = eg_E(v),$$

then, for every bounded Lipschitz Ω and every $w - v \in W_0^{1,2}(\Omega)$,

$$(2.9) \quad \mathcal{A}_b[w; \Omega] - \mathcal{A}_b[v; \Omega] = \frac{1}{2} \int_{\Omega} |\nabla w - eg_E(w)|^2 dx \geq 0.$$

Moreover v solves (2.6).

Proof. The chain rule gives $eg_E(w) \cdot \nabla w = e \cdot \nabla G_E(w)$. The function $G_E(w) - G_E(v)$ has zero boundary trace. Therefore

$$\int_{\Omega} e \cdot \nabla (G_E(w) - G_E(v)) dx = 0.$$

Subtract the integrated identities (2.7) for w and v . The square for v vanishes, which proves (2.9). No restriction on the dependence of w on the transverse variables was made.

For $E > 0$, differentiation of (2.8) gives

$$\Delta v = g_E(v)g'_E(v) = b(v)b'(v).$$

For $E = 0$ the same calculation reads $\Delta v = b(v)b'(v)$ directly. Thus the equation also holds at the zeros of b . $\square$

There is an equivalent differential-form calibration. Let $dx = dx_1 \wedge \cdots \wedge dx_n$ and $dx_i^\# = \iota_{\partial x_i} dx$. Then

$$(2.10) \quad \omega_{E,e} = \sum_{i=1}^n e_i g_E(u) du \wedge dx_i^\# - E dx$$

is a smooth closed periodic n -form. Its pullback to the graph of w is $(eg_E(w) \cdot \nabla w - E)dx$. Equation (2.7) is precisely the calibration inequality for this form. The direct Sobolev proof above avoids any need to interpret Stokes' theorem on nonsmooth graph chains.

2.2. *Every nonzero slope.* For $E > 0$ define

$$(2.11) \quad T(E) = \int_0^1 \frac{ds}{\sqrt{b(s)^2 + 2E}}.$$

Differentiation under the integral gives

$$T'(E) = - \int_0^1 \frac{ds}{(b(s)^2 + 2E)^{3/2}} < 0.$$

Also $T(E) \leq (2E)^{-1/2}$, so $T(E) \rightarrow 0$ as $E \rightarrow \infty$.

Let s_0 be a zero of b . Because b is Lipschitz, $b(s) \leq C|s - s_0|$ near s_0 . Hence $1/b$ has a nonintegrable singularity there; if b vanishes on an interval, the conclusion is even immediate. Monotone convergence in (2.11) yields $T(E) \rightarrow \infty$ as $E \downarrow 0$. Thus, for every $r > 0$, there is exactly one $E(r) > 0$ such that

$$(2.12) \quad T(E(r)) = \frac{1}{r}.$$

Fix $\alpha \neq 0$, set $r = |\alpha|$ and $e = \alpha/r$, and define

$$(2.13) \quad \vartheta_E(s) = r \int_0^s \frac{da}{g_E(a)}, \quad f_E = \vartheta_E^{-1}.$$

Equation (2.12) gives

$$\vartheta_E(s+1) = \vartheta_E(s) + 1, \quad f_E(z+1) = f_E(z) + 1, \quad f'_E(z) = \frac{g_E(f_E(z))}{r} > 0.$$

Now put

$$(2.14) \quad U_\alpha(x, t) = f_E(t + \alpha \cdot x).$$

Then

$$\nabla_x U_\alpha = \alpha f'_E(t + \alpha \cdot x) = eg_E(U_\alpha).$$

Lemma 2.4 proves the equation and class-A minimality of every leaf. The map in (2.14) is smooth and strictly increasing in t , has vertical degree one, and satisfies

$$U_\alpha(x + m, t) = U_\alpha(x, t + \alpha \cdot m).$$

Since $f_E(z) - z$ is periodic and bounded, $U_\alpha(x, t) - \alpha \cdot x$ is bounded. This verifies every part of Definition 2.2, together with rigid holonomy.

2.3. *The zero-slope foliation.* Let Φ_s denote the flow of

$$(2.15) \quad \dot{y} = b(y).$$

The vector field and its derivatives are bounded. Thus Φ_s is defined for every real s , is smooth, and satisfies

$$(2.16) \quad \Phi_{s+r} = \Phi_s \circ \Phi_r, \quad \Phi_s(t+1) = \Phi_s(t) + 1,$$

$$(2.17) \quad \partial_t \Phi_s(t) = \exp \left(\int_0^s b'(\Phi_a(t)) da \right) > 0.$$

Every orbit remains between two consecutive zeros of the periodically extended zero set, or is itself constant. In particular, its displacement is bounded. Define

$$(2.18) \quad U_0(x, t) = \Phi_{x_1}(t).$$

It satisfies $\nabla_x U_0 = e_1 b(U_0)$, so Lemma 2.4 again proves the equation and class-A minimality. Boundedness of the flow orbits gives slope zero. The deck action is

$$(2.19) \quad U_0(x + m, t) = U_0(x, \Phi_{m_1}(t)), \quad F_{0,m} = \Phi_{m_1}.$$

At a point where $b(t) > 0$, $\Phi_1(t) > t$. At every zero of b , $\Phi_1(t) = t$. Consequently Φ_1 is not the identity, although it has translation number zero.

The descended distribution is spanned by

$$\partial_{x_1} + b(u)\partial_u, \quad \partial_{x_2}, \dots, \partial_{x_n}.$$

These are smooth independent commuting vector fields on M . The foliation is therefore genuinely smooth, including along its constant leaves. Degeneration of transverse derivatives as $x_1 \rightarrow \infty$ in a particular lifted parametrization does not create a singularity of the torus foliation.

2.4. *The explicit model and its intrinsic holonomy obstruction.* Take

$$(2.20) \quad b(u) = \sin^2(\pi u), \quad V_*(u) = -\frac{1}{2} \sin^4(\pi u).$$

Here $V'_*(1/4) = -\pi/2 \neq 0$. On each interval $(j, j+1)$,

$$(2.21) \quad \cot(\pi \Phi_s(t)) = \cot(\pi t) - \pi s.$$

The nonconstant zero-slope leaves can equivalently be written as

$$(2.22) \quad v_{j,a}(x) = j + \frac{1}{2} + \frac{1}{\pi} \arctan(\pi(x_1 - a)), \quad a \in \mathbb{R}.$$

They connect the consecutive constant levels j and $j+1$. In the torus those integer levels project to a single compact leaf, and the remaining leaves are noncompact. Relabelling conjugates $F_{0,e_1} = \Phi_1$, so it cannot change this nonidentity map into the identity.

One must also rule out a *different* zero-slope minimizing foliation with identity holonomy. The following elementary lemma does so.

LEMMA 2.5 (Periodic class-A minimizers of a nonnegative autonomous action). *Let $W \in C^\infty(\mathbb{T})$ satisfy $W \geq 0$ and $W(s_*) = 0$ for some s_* . If a smooth function w is periodic with respect to a full-rank lattice $\Lambda \subset \mathbb{R}^n$ and is a class-A minimizer of*

$$\int \left(\frac{1}{2} |\nabla w|^2 + W(w) \right) dx,$$

then w is constant and $W(w) = 0$.

Proof. Let D be a fundamental parallelepiped and set

$$\delta = \frac{1}{|D|} \int_D \left(\frac{1}{2} |\nabla w|^2 + W(w) \right) dx.$$

The density is periodic and nonnegative. For $C_R = (-R, R)^n$, tiling by lattice cells gives

$$\int_{C_R} \left(\frac{1}{2} |\nabla w|^2 + W(w) \right) dx = \delta |C_R| + O(R^{n-1}).$$

To see the error estimate, only cells meeting a fixed-width neighborhood of ∂C_R are unaccounted for; their total volume is $O(R^{n-1})$, and the density is bounded.

Choose a smooth cutoff η_R equal to one on C_{R-1} , zero near ∂C_R , with $|\nabla \eta_R| \leq C$ independent of R . The competitor

$$w_R = (1 - \eta_R)w + \eta_R s_*$$

agrees with w near the boundary and is constant at a zero of W on C_{R-1} . Periodicity bounds w and ∇w uniformly, so the energy of w_R is at most CR^{n-1} . If $\delta > 0$, this contradicts class-A minimality for large R . Thus $\delta = 0$. Nonnegativity and continuity of the density imply $\nabla w = 0$ and $W(w) = 0$ everywhere. $\square$

For (2.20), the only periodic class-A minimizers on any full-rank lattice are the integer constants. A zero-slope identity-holonomy foliation would consist of $\mathbb{Z}^n$ -periodic class-A minimizers, and these cannot fill the vertical circle. This proves the zero-slope obstruction in Theorem 2.3. The rational relocation is proved in Section 3.

The same argument applies to every b in Theorem 2.3, not only to the trigonometric profile. A periodic minimizing leaf is constant with value in the proper closed set $\{b = 0\}$. Such leaves cannot fill the vertical circle. Hence there is no alternative closed zero-slope minimizing foliation for any member of the stated class.

3. Placing the obstruction at an arbitrary rational slope

The missing holonomy condition cannot be imposed only at slope zero. The obstruction can be moved to any rational slope without changing the Euclidean kinetic energy except for a multiplicative factor and a null Lagrangian.

3.1. *The intrinsic rational condition.* Fix $\alpha \in \mathbb{Q}^n$ and write $\alpha = -k/\ell$, with $k \in \mathbb{Z}^n$ and $\ell \in \mathbb{N}$. Set

$$(3.1) \quad \Lambda_\alpha = \{m \in \mathbb{Z}^n : \alpha \cdot m \in \mathbb{Z}\} = \{m \in \mathbb{Z}^n : k \cdot m \equiv 0 \pmod{\ell}\}.$$

This lattice does not depend on the choice of the representation of α .

Definition 3.1 (Identity adjusted rational holonomy). An ordinary foliation of rational slope α has identity adjusted rational holonomy if

$$(3.2) \quad U(x + \lambda, t) = U(x, t) + \alpha \cdot \lambda \quad \text{for every } \lambda \in \Lambda_\alpha.$$

Thus each leaf closes with its prescribed vertical increments on the natural finite cover. This condition is invariant under every degree-one relabelling of the leaves.

In an arbitrary leaf parameter, (3.2) is the identity $R_{-\alpha \cdot \lambda} \circ F_\lambda = \text{Id}$, where $R_a(t) = t + a$. One should not replace it by a claim that all the F_m are already translations in that arbitrary parameter.

LEMMA 3.2 (Rigid normalization of rational holonomy). *A smooth ordinary foliation of rational slope α satisfies (3.2) if and only if some increasing degree-one smooth relabelling gives*

$$(3.3) \quad \widetilde{U}(x + m, t) = \widetilde{U}(x, t + \alpha \cdot m) \quad (m \in \mathbb{Z}^n).$$

Proof. The reverse implication follows from vertical degree one. For the forward implication, reduce the commuting maps F_m modulo $\mathbb{Z}$. Equation (3.2) implies that their action factors through the finite group $G = \mathbb{Z}^n / \Lambda_\alpha$. Let f_g denote its circle diffeomorphisms, and define the positive smooth periodic density

$$a(t) = \frac{1}{|G|} \sum_{g \in G} f'_g(t).$$

Derivatives are independent of the choices of degree-one lifts. Each summand integrates to one, so $\int_0^1 a = 1$. Composition permutes G , and the chain rule gives

$$a(F_m(t))F'_m(t) = a(t).$$

Hence $h(t) = \int_0^t a(s) ds$ is an increasing degree-one lift for which $h(F_m(t)) - h(t) = c_m$ is constant in t .

The constants are determined by the slopes, rather than by a choice of representatives of G . Iterating (2.3) at $x = 0$ gives

$$U(Nm, t) = U(0, F_m^N(t)).$$

The function $U(0, s) - s$ is bounded and $U(Nm, t) - N\alpha \cdot m$ is bounded as $N \rightarrow \infty$. Therefore $F_m^N(t)/N \rightarrow \alpha \cdot m$. Since $h(s) - s$ is bounded, the equality $h(F_m^N(t)) = h(t) + Nc_m$ yields $c_m = \alpha \cdot m$. Taking $\widetilde{U}(x, t) = U(x, h^{-1}(t))$ proves (3.3). $\square$

3.2. The rational shear and its variational invariance.

THEOREM 3.3 (An arbitrarily prescribed exceptional rational slope). *For $k \in \mathbb{Z}^n$ and $\ell \in \mathbb{N}$, let*

$$(3.4) \quad V_{k,\ell}(x, u) = -\frac{1}{2\ell^2} \sin^4(\pi(\ell u + k \cdot x)), \quad \alpha_0 = -\frac{k}{\ell}.$$

Then $(V_{k,\ell})_u \not\equiv 0$. The potential admits an ordinary smooth minimizing foliation at every slope. Every slope $\alpha \neq \alpha_0$ admits a rigid minimizing foliation. At α_0 , no minimizing foliation has identity adjusted rational holonomy.

Proof. Set $\theta = \ell v + k \cdot x$. A direct expansion gives

$$(3.5) \quad \mathcal{A}_{V_{k,\ell}}[v; \Omega] = \frac{1}{\ell^2} \mathcal{A}_{V_*}[\theta; \Omega] - \frac{1}{\ell^2} \int_{\Omega} k \cdot \nabla \theta \, dx + \frac{|k|^2}{2\ell^2} |\Omega|.$$

For competitors with the same boundary trace, the last two terms have the same value. The correspondence $w \mapsto \ell w + k \cdot x$ is a bijection on the corresponding fixed-trace competitor classes. Thus class-A minimality is equivalent under this transformation, with no restriction on the dimension or on competitor dependence.

For a specified slope α , put $\beta = \ell\alpha + k$. If $\beta \neq 0$, use the profile of Section 2 for slope β and define

$$(3.6) \quad U_{\alpha}(x, t) = \frac{f_E(|\beta|)(\ell t + \beta \cdot x) - k \cdot x}{\ell}.$$

It is strictly increasing in t and has vertical degree one. Its leaves are minimizing by (3.5), and $U_{\alpha}(x, t) - \alpha \cdot x$ is bounded. For $m \in \mathbb{Z}^n$, the number $k \cdot m$ is integral, so

$$\begin{aligned} U_{\alpha}(x + m, t) &= \frac{f_E(\ell t + \beta \cdot x + \ell\alpha \cdot m + k \cdot m) - k \cdot x - k \cdot m}{\ell} \\ &= U_{\alpha}(x, t + \alpha \cdot m). \end{aligned}$$

This is the rigid deck law.

When $\beta = 0$, let Φ be the flow of $\sin^2(\pi u)$ and set

$$(3.7) \quad U_{\alpha_0}(x, t) = \frac{\Phi_{x_1}(\ell t) - k \cdot x}{\ell}.$$

The same action identity proves minimality. Boundedness of $\Phi_{x_1}(\ell t)$ proves slope α_0 . The deck maps are

$$(3.8) \quad F_m(t) = \frac{\Phi_{m_1}(\ell t) - k \cdot m}{\ell}.$$

Indeed, substitution of this expression in (3.7), using the flow law and integer vertical translations, gives $U(x+m, t) = U(x, F_m(t))$. These maps are increasing degree-one lifts. For $m = \ell e_1 \in \Lambda_{\alpha_0}$, the adjusted return is not the identity because $\Phi_\ell \neq \text{Id}$.

Finally, suppose a different minimizing foliation at α_0 satisfied (3.2). Each transformed leaf

$$\theta_t(x) = \ell U(x, t) + k \cdot x$$

would be Λ_{α_0} -periodic and a class-A minimizer for V_* . Lemma 2.5 forces θ_t to be an integer constant. Such leaves cannot form a strictly ordered continuous family covering all vertical values. This excludes every identity-holonomy foliation at the exceptional slope, not only the particular family (3.7). $\square$

COROLLARY 3.4 (Sharpness of the rational testing set). *If $S \subset \mathbb{R}^n$ omits a rational vector, a smooth potential with $V_u \not\equiv 0$ has ordinary minimizing foliations at every slope and rigid minimizing foliations at every slope in S .*

Proof. Choose $\alpha_0 \in \mathbb{Q}^n \setminus S$ and apply Theorem 3.3. $\square$

We write $\text{RH}_{\mathbb{Q}}$ for the requirement that every rational slope admit a minimizing foliation satisfying (3.2). The corollary proves the necessity of testing every rational slope, not the sufficiency of $\text{RH}_{\mathbb{Q}}$ for rigidity.

4. Invariant volume, closed phases, and calibration

Definition 4.1 (Common invariant volume). For a foliation field p_a , put

$$(4.1) \quad D_i^a = \partial_{x_i} + p_{a,i} \partial_u.$$

A common positive invariant probability volume for a selected collection is one $q \in C^1(M)$, $q > 0$, $\int_M q \, dm = 1$, such that

$$(4.2) \quad \partial_{x_i} q + \partial_u (q p_{a,i}) = 0 \quad (a \in S, 1 \leq i \leq n).$$

Equivalently, each of the indicated horizontal fields preserves $q \, dm$.

4.1. The fixed-torus equations. Let p be the smooth periodic slope field of an ordinary graph foliation by solutions. Equality of mixed derivatives on a leaf gives

$$(4.3) \quad \partial_{x_i} p_j + p_i \partial_u p_j = \partial_{x_j} p_i + p_j \partial_u p_i.$$

The equation on every leaf gives

$$(4.4) \quad \mathcal{R}(p) := \operatorname{div}_x p + \partial_u \left(\frac{1}{2} |p|^2 \right) = Q.$$

Both assertions are pointwise identities on M ; no derivative with respect to slope is involved.

Definition 4.2 (Admissible solution phase). An admissible phase at slope α is

$$(4.5) \quad \tau(x, u) = u - \alpha \cdot x + \psi(x, u), \quad \psi \in C^2(M), \quad q := \tau_u > 0,$$

whose field $p = -\nabla_x \tau / q$ satisfies (4.4).

Its inverse $u = U(x, t)$ satisfies

$$(4.6) \quad U_t = \frac{1}{q(x, U)}, \quad \nabla_x U = p(x, U), \quad \Delta_x U = Q(x, U).$$

The periodic correction gives the rigid deck law (3.3) and bounded deviation from $\alpha \cdot x$.

LEMMA 4.3 (Common volume is equivalent to simultaneous phase density). *A positive C^1 probability volume $q \, dm$ is invariant under every horizontal field of an ordinary solution graph foliation of slope α if and only if that foliation admits an admissible phase with $\tau_u = q$. In particular,*

$$(4.7) \quad \int_0^1 q(x, u) \, du = 1, \quad \int_M q p \, dm = \alpha.$$

One volume works for a collection of foliations if and only if their phases can be chosen with the same density q .

Proof. Invariance is the equation

$$(4.8) \quad \partial_{x_i} q + \partial_u (q p_i) = 0.$$

Integration in u makes $\int_0^1 q(x, u) \, du$ independent of x . Its integral over $\mathbb{T}^n$ is one, so it equals one pointwise.

Consider the one-form

$$\theta = q \left(du - \sum_i p_i dx_i \right).$$

Its mixed x_i, u components are closed by (4.8). For its horizontal components, use (4.8) to calculate

$$\partial_{x_i} (q p_j) - \partial_{x_j} (q p_i) = q (\partial_{x_i} p_j - \partial_{x_j} p_i + p_i p_{j,u} - p_j p_{i,u}) = 0$$

by (4.3). Thus $d\theta = 0$.

The vertical period of θ is one. The cohomology of the torus therefore gives a vector $a \in \mathbb{R}^n$ and a C^2 periodic function ψ with

$$\theta = du - a \cdot dx + d\psi.$$

On the universal cover it is $d\tau$ for $\tau = u - a \cdot x + \psi$. Positivity of q makes the level sets graphs. Their horizontal tangent distribution is the original one, so these are the same leaves. Since ψ is bounded, their slope is a . The prescribed slope of the foliation is α , hence $a = \alpha$. This proves the phase representation and (4.7). The converse follows by taking mixed derivatives of (4.5). $\square$

Remark 4.4. At a rational slope, replacing τ by an admissible $h \circ \tau$ changes q to $h'(\tau)q$ without changing p . Common volume means that compatible choices exist simultaneously; it does not mean that arbitrary individually chosen phase densities agree. Conversely, the invariant-volume formulation itself is unaffected by relabelling.

4.2. *Minimality requires no additional assumption.*

LEMMA 4.5 (Periodic Hilbert calibration). *Let $p \in C^\infty(M; \mathbb{R}^n)$ satisfy (4.4). Then*

$$(4.9) \quad \omega_p = \sum_{i=1}^n p_i du \wedge dx_i^\# - \left(\frac{1}{2}|p|^2 + V \right) dx$$

is closed. Every smooth entire graph v satisfying $\nabla v = p(x, v)$ is class-A minimizing. For every fixed-trace competitor w , one has the exact identity

$$(4.10) \quad \mathcal{A}_V[w; \Omega] - \mathcal{A}_V[v; \Omega] = \frac{1}{2} \int_{\Omega} |\nabla w - p(x, w)|^2 dx.$$

Proof. A direct calculation gives

$$d\omega_p = - \left(\operatorname{div}_x p + \partial_u \left(\frac{1}{2}|p|^2 + V \right) \right) du \wedge dx = 0.$$

On the graph of any smooth w ,

$$(4.11) \quad \left(\frac{1}{2}|\nabla w|^2 - V(x, w) \right) dx = w^* \omega_p + \frac{1}{2} |\nabla w - p(x, w)|^2 dx.$$

For completeness, the invariance of the first term under fixed-trace changes can be proved without using graph chains. Set

$$N[w] = \int_{\Omega} \left(p(x, w) \cdot \nabla w - \frac{1}{2}|p(x, w)|^2 - V(x, w) \right) dx.$$

For a smooth compactly supported variation η , differentiation and integration by parts give

$$\begin{aligned} DN[w]\eta &= \int_{\Omega} (p_u(x, w) \cdot \nabla w \eta + p(x, w) \cdot \nabla \eta) dx \\ &\quad - \int_{\Omega} (p(x, w) \cdot p_u(x, w) + V_u(x, w)) \eta dx \\ &= - \int_{\Omega} (\operatorname{div}_x p + p \cdot p_u + V_u)(x, w) \eta dx = 0. \end{aligned}$$

Integrating along $w_s = v + s\eta$ shows $N[v + \eta] = N[v]$. Appendix J extends this equality to every $w - v \in W_0^{1,2}(\Omega)$. Subtract (4.11) for w and v and use $\nabla v = p(x, v)$. $\square$

5. Phase locking and rational spheres

The decisive calculation compares two genuine foliations with the same density. First the periodic horizontal error is eliminated. The remaining equation is a quadratic identity in their slope difference.

LEMMA 5.1 (Phase locking). *Let $\tau_a = u - a \cdot x + \psi_a$ and $\tau_b = u - b \cdot x + \psi_b$ be admissible phases for the same V , with*

$$(\tau_a)_u = (\tau_b)_u = q > 0.$$

Then $\psi_b - \psi_a$ is constant and

$$(5.1) \quad p_b - p_a = \frac{b - a}{q}.$$

Consequently the common invariant probability volume of two distinct-slope foliations, when it exists, is unique.

Proof. Set $g = \psi_b - \psi_a$. Equality of the vertical derivatives makes g a function of x alone, periodic on $\mathbb{T}^n$. Write

$$h = b - a, \quad A(x) = h - \nabla g(x), \quad p = p_a, \quad r = p_u.$$

Then $p_b = p + A/q$. The continuity equation for p reads

$$(5.2) \quad \nabla_x q = -pq_u - qr.$$

Expand the residual difference, keeping A independent of u :

$$\begin{aligned} 0 &= \mathcal{R}(p + A/q) - \mathcal{R}(p) \\ &= \operatorname{div}_x(A/q) + \partial_u \left(\frac{p \cdot A}{q} + \frac{|A|^2}{2q^2} \right) \\ (5.3) \quad &= \frac{\operatorname{div}_x A}{q} + \frac{2A \cdot r}{q} - \frac{|A|^2 q_u}{q^3}. \end{aligned}$$

The cancellation in the last line follows directly from (5.2). Multiply by q and integrate over one u -period. Since p and q are periodic,

$$\int_0^1 r \, du = 0, \quad \int_0^1 \frac{q_u}{q^2} \, du = - \int_0^1 \partial_u(q^{-1}) \, du = 0.$$

It follows that $\operatorname{div}_x A = 0$, or $\Delta g = 0$. Multiplying by g and integrating on $\mathbb{T}^n$ gives $\nabla g = 0$. This proves (5.1).

If $a \neq b$, a coordinate with $(b - a)_i \neq 0$ gives

$$q = \frac{(b - a)_i}{(p_b - p_a)_i}.$$

Thus at most one normalized density can be common to these two fixed foliations. The denominator is nowhere zero when a common positive density exists. $\square$

PROPOSITION 5.2 (The pointwise slope identity). *With the hypotheses of Lemma 5.1, let $h = b - a$. Then*

$$(5.4) \quad 2q^2 h \cdot \partial_u p_a = |h|^2 q_u \quad \text{on } M.$$

Proof. After phase locking, $A = h$ is constant in (5.3). Multiply the remaining equation by q^3 . $\square$

LEMMA 5.3 (The flat density forces a flat source). *If an admissible phase has $q_u = 0$, then $q = 1$, $Q = 0$, and its slope field is constant.*

Proof. Equation (4.7) gives $q = 1$. Equation (4.8) gives $p_u = 0$, so (4.4) makes Q independent of u . Its vertical mean is zero by (2.2); therefore $Q = 0$. Now $\tau = u - \alpha \cdot x + \psi(x)$ and $p = \alpha - \nabla \psi$. The equation says $\Delta \psi = 0$, and periodicity gives $p = \alpha$. $\square$

5.1. *Removing the spatial mean.* The pointwise identity contains a further arithmetic restriction which is not visible if it is evaluated at a single point. It is important to retain its full dependence on (x, u) .

For a collection at slopes S with a common volume, Lemma 5.1 allows additive constants in the phases to be chosen so that

$$(5.5) \quad \tau_a = u - a \cdot x + \psi(x, u), \quad q = 1 + \psi_u, \quad p_a = \frac{a - \nabla_x \psi}{q} \quad (a \in S).$$

If two slopes are distinct, the uniqueness formula in Lemma 5.1 also makes q smooth, since the slope fields are smooth. Thus the common ψ can be taken smooth. This observation supplies the smoothness used in the classification below; it is not an extra transverse assumption.

LEMMA 5.4 (Mean-removing shear). *Fix $a \in S$ and put*

$$(5.6) \quad f(x) = \int_0^1 \psi(x, u) \, du, \quad s = u + f(x), \quad \chi(x, s) = \psi(x, s - f(x)) - f(x).$$

Then χ is periodic in (x, s) ,

$$(5.7) \quad \int_0^1 \chi(x, s) ds = 0, \quad q(x, u) = 1 + \chi_s(x, u + f(x)) > 0,$$

and, for every $b \in S$,

$$(5.8) \quad h \cdot \nabla_x \chi + \delta \chi_s = 0, \quad h = b - a, \quad \delta = \frac{|b|^2 - |a|^2}{2}.$$

Here $\nabla_x \chi$ is taken with s fixed.

Proof. Since $q_u = \psi_{uu}$ and $\nabla_x q = \nabla_x \psi_u$, substitution of (5.5) in (5.4) gives

$$|h|^2 q_u = 2q^2 h \cdot \left(-\frac{\nabla_x q}{q} - \frac{(a - \nabla_x \psi) q_u}{q^2} \right),$$

so

$$(5.9) \quad q h \cdot \nabla_x q = (h \cdot \nabla_x \psi - \delta) q_u, \quad \partial_u \left(\frac{h \cdot \nabla_x \psi - \delta}{q} \right) = 0.$$

Write the u -independent quotient as $c_h(x)$. Integrating $h \cdot \nabla_x \psi - \delta = c_h q$ over one physical vertical period and using $\int_0^1 q du = 1$ gives

$$c_h(x) = h \cdot \nabla f(x) - \delta.$$

Therefore

$$(5.10) \quad h \cdot \nabla_x (\psi - f) = (\psi - f)_u (h \cdot \nabla f - \delta).$$

The map $(x, u) \mapsto (x, u + f(x))$ is a periodic diffeomorphism of M , with inverse $(x, s) \mapsto (x, s - f(x))$. Under this map,

$$\nabla_x (\psi - f) = \nabla_x \chi + \chi_s \nabla f, \quad (\psi - f)_u = \chi_s.$$

Substitution in (5.10) cancels the terms containing $h \cdot \nabla f$ and yields (5.8). Periodicity and the zero mean in (5.7) follow directly from (5.6). The identity for q follows by differentiating in u . $\square$

The shear is used to rewrite an exact compatibility equation. We do not assert that it preserves the Euclidean kinetic action. Its effect on the source will be calculated separately in Section 7.

THEOREM 5.5 (Arithmetic spherical alternative). *Let S contain at least two distinct slopes of solution foliations preserving a common positive invariant volume. If $V_u \neq 0$, then*

$$\mathcal{C}_{\mathbb{Q}}(S) \neq \emptyset.$$

In particular, all slopes in the collection lie on one sphere with rational centre, whether or not their affine span is the whole base.

Proof. Use the representation in Lemma 5.4. If $\chi = 0$, then $q = 1$, and Lemma 5.3 gives $V_u = 0$. Thus $\chi \neq 0$ on the nontrivial branch.

For $(k, m) \in \mathbb{Z}^n \times \mathbb{Z}$, set

$$\widehat{\chi}(k, m) = \int_{\mathbb{T}^n \times \mathbb{T}} \chi(x, s) e^{-2\pi i(k \cdot x + ms)} dx ds.$$

The zero vertical mean gives $\widehat{\chi}(k, 0) = 0$ for every k . Fourier uniqueness therefore supplies a coefficient $\widehat{\chi}(k, m) \neq 0$ with $m \neq 0$. For this one coefficient, (5.8) gives

$$\left((b - a) \cdot k + \frac{|b|^2 - |a|^2}{2} m \right) \widehat{\chi}(k, m) = 0 \quad \text{for every } b \in S.$$

It follows that $\beta = -k/m \in \mathbb{Q}^n$ satisfies

$$(b - a) \cdot \beta = \frac{|b|^2 - |a|^2}{2} \quad (b \in S).$$

These are exactly the equations defining $\mathcal{C}_{\mathbb{Q}}(S)$. A single nonzero Fourier coefficient works for the entire collection, including an infinite collection. The Fourier calculation is on the genuinely periodic function χ , not on a phase character which might fail to descend at an irrational slope. $\square$

6. The exact rigidity criterion and finite tests

We can now state the exact criterion. The preceding phase calculation proves the rigidity direction. The counterexample gives the converse while retaining minimizing foliations at every other slope. For a nonempty set $S \subset \mathbb{R}^n$, define its rational centre set by

$$(6.1) \quad \mathcal{C}_{\mathbb{Q}}(S) = \{ \beta \in \mathbb{Q}^n : |a - \beta|^2 = |b - \beta|^2 \text{ for all } a, b \in S \}.$$

Thus $\mathcal{C}_{\mathbb{Q}}(S) \neq \emptyset$ means that S lies on a Euclidean sphere with rational centre. When S contains at least two points, its radius is positive. For a fixed $a \in S$, the condition is the linear system

$$(6.2) \quad (b - a) \cdot \beta = \frac{|b|^2 - |a|^2}{2} \quad (b \in S), \quad \beta \in \mathbb{Q}^n.$$

THEOREM 6.1 (Exact common-volume rigidity criterion). *Fix a set $S \subset \mathbb{R}^n$ containing at least two distinct slopes. The following are equivalent.*

- (i) $\mathcal{C}_{\mathbb{Q}}(S) = \emptyset$.
- (ii) *Every smooth periodic potential admitting solution foliations at the slopes in S with one common positive C^1 invariant probability volume satisfies $V_u = 0$.*
- (iii) *The conclusion in (ii) holds even when the class of potentials is restricted to those having ordinary smooth minimizing foliations at every real slope.*

If $\mathcal{C}_{\mathbb{Q}}(S) = \emptyset$, the common density is $q = 1$ and the selected foliations are affine. If $\mathcal{C}_{\mathbb{Q}}(S) \neq \emptyset$, a potential of the form (2.5) gives a counterexample to (ii) and (iii). Its selected foliations are rigid and share a common volume.

The positive proof uses only the selected slopes. The all-slope assumption is retained in the converse to make the optimality statement relative to the original problem, rather than to a larger class of unrelated examples.

6.1. *The converse keeps the all-slope hypothesis.*

PROPOSITION 6.2 (Every rational-centred sphere is realized). *Let $\beta = -k/\ell \in \mathbb{Q}^n$, with $\ell \in \mathbb{N}$, and let $R > 0$. There is a nonconstant smooth periodic potential with ordinary minimizing foliations at every slope such that all slopes on*

$$|\alpha - \beta| = R$$

have rigid minimizing foliations preserving one common smooth positive probability volume.

Proof. Use the rational-shear potential (3.4), with $b(z) = \sin^2(\pi z)$ and $z = \ell u + k \cdot x$. Choose $E > 0$ by

$$\int_0^1 \frac{dz}{\sqrt{b(z)^2 + 2E}} = \frac{1}{\ell R}.$$

Write $g_E = \sqrt{b^2 + 2E}$ and let ϑ_E be the degree-one primitive with $\vartheta'_E = \ell R/g_E$. Define

$$(6.3) \quad q(x, u) = \frac{\ell R}{g_E(z)}, \quad \tau_\alpha(x, u) = \frac{\vartheta_E(z)}{\ell} - (\alpha - \beta) \cdot x.$$

Because $\vartheta_E(z) - z$ is periodic and k, ℓ are integral,

$$\tau_\alpha = u - \alpha \cdot x + \frac{\vartheta_E(z) - z}{\ell}$$

has the required cohomology. Also $\partial_u \tau_\alpha = q > 0$ and

$$(6.4) \quad p_\alpha = \beta + \frac{\alpha - \beta}{q}.$$

The fibre integral of q is one by the defining equation for E . These phases are exactly the nonzero-slope calibrated phases from the autonomous construction after the rational shear: the transformed slope is $\ell(\alpha - \beta)$, of length ℓR . Thus their fields solve the equation and their leaves are class-A minimizing. Alternatively,

$$V_{k,\ell} = \frac{E}{\ell^2} - \frac{R^2}{2q^2}, \quad \nabla_x q + \beta q_u = 0,$$

verify the residual equation and invariance directly. Theorem 3.3 supplies ordinary minimizing foliations at all other slopes, independently of this selected sphere. The potential is nonconstant. $\square$

Proof of Theorem 6.1. If $\mathcal{C}_{\mathbb{Q}}(S) = \emptyset$, Theorem 5.5 gives $V_u = 0$. More directly, its Fourier proof makes $\chi = 0$, so Lemma 5.3 gives $q = 1$ and $p_a = a$ for every selected slope. This proves (i) $\Rightarrow$ (ii). Restricting the class of potentials proves (ii) $\Rightarrow$ (iii).

If (i) fails, choose $\beta \in \mathcal{C}_{\mathbb{Q}}(S)$. The common radius $R = |a - \beta|$ is positive because S contains two distinct points. Proposition 6.2 supplies a nontrivial counterexample even in the restricted all-slope class of (iii). Thus (iii) $\Rightarrow$ (i), completing all directions. $\square$

Theorem 6.1 gives a classification, not just an example showing that one hypothesis cannot be deleted. We now record the resulting minimality statements with their quantifiers explicit.

6.2. Two real slopes and three rational slopes.

PROPOSITION 6.3 (Optimal cardinalities). *Among tests consisting of prescribed solution foliations with a common positive invariant volume, the smallest possible number of slopes is two. Among such tests restricted to rational slopes, the smallest possible number is three.*

Proof. For one prescribed slope a , choose a rational vector $\beta \neq a$ and set $R = |a - \beta|$. Proposition 6.2 gives a nontrivial all-slope example whose a -foliation preserves a positive volume. Thus one never suffices.

For two distinct slopes a, b , Theorem 6.1 says that rigidity holds exactly when their perpendicular bisector

$$(6.5) \quad (b - a) \cdot \beta = \frac{|b|^2 - |a|^2}{2}$$

contains no rational vector. Take $a = 0$ and $b = \sqrt{2}e_1$. Equation (6.5) would require $\beta_1 = 1/\sqrt{2}$, which is impossible for $\beta \in \mathbb{Q}^n$. These two slopes therefore suffice.

If $a, b \in \mathbb{Q}^n$, their midpoint $(a + b)/2$ is a rational centre. Hence no pair of distinct rational slopes suffices. On the other hand, a Euclidean sphere meets a line in at most two points. Any three distinct collinear slopes have empty rational centre set; in particular $\{-e_1, 0, e_1\}$ is a sufficient rational test. This proves both minima. $\square$

The assertion that two foliations can fail must not be read as saying that every pair fails. The distinction is arithmetic. Arbitrarily many noncollinear slopes on one rational-centred sphere still fail, whereas a suitable pair with irrational separation succeeds.

For example, if $a = te_1$ and $b = se_1$ with $s \neq t$, then (6.5) requires $\beta_1 = (s + t)/2$. This pair is rigid exactly when $(s + t)/2$ is irrational. If both slopes are rational, the arithmetic obstruction disappears and a third point on the same line is required.

6.3. Rational tests are exactly the noncospherical tests.

PROPOSITION 6.4. *For $S \subset \mathbb{Q}^n$, the set $\mathcal{C}_{\mathbb{Q}}(S)$ is empty if and only if no Euclidean sphere contains S .*

Proof. Only the converse implication needs proof. If a sphere contains S , its centre solves the real linear equations (6.2), whose coefficients and right sides are rational. Select finitely many independent rows spanning all row vectors. Consistency makes every remaining equation a consequence of them. Gaussian elimination over $\mathbb{Q}$ gives a rational particular solution, so $\mathcal{C}_{\mathbb{Q}}(S) \neq \emptyset$. This also handles infinite S , since the row rank is at most n . $\square$

Thus a condition of not lying on any sphere is sufficient for all real slope sets, but is not necessary there. The exact condition over real slopes is the absence of a *rational* centre.

6.4. Finite witnesses and deletion-minimality.

Definition 6.5. A finite set S is an inclusion-minimal common-volume rigidity test if it is sufficient in Theorem 6.1 and no proper subset is sufficient. Singletons are understood to be insufficient, as in Proposition 6.3.

THEOREM 6.6 (Finite-witness theorem). *If $\mathcal{C}_{\mathbb{Q}}(S) = \emptyset$, there is a subset $S_0 \subset S$ with $|S_0| \leq n + 2$ and $\mathcal{C}_{\mathbb{Q}}(S_0) = \emptyset$. Moreover, a finite set is inclusion-minimal exactly when*

$$(6.6) \quad \mathcal{C}_{\mathbb{Q}}(S) = \emptyset, \quad \mathcal{C}_{\mathbb{Q}}(S \setminus \{a\}) \neq \emptyset \quad (a \in S).$$

The bound $n + 2$ is attained by rational slope sets.

Proof. Fix $a \in S$ and write the centre equations as $h_b \cdot \beta = \delta_b$, where $h_b = b - a$. Choose $r \leq n$ independent row vectors $h_{b_1}, \dots, h_{b_r}$ spanning all the h_b .

Suppose first that the entire system is consistent over $\mathbb{R}$. Then every remaining equation has the same right side as the corresponding linear combination of these r equations. The set of real solutions is therefore already determined by $a, b_1, \dots, b_r$. Since this solution set contains no rational vector, these at most $n + 1$ slopes give the required witness.

Suppose instead that the system is inconsistent over $\mathbb{R}$. The r independent equations alone are consistent. At least one additional row, say that of b_* , has a right side different from its forced linear combination. Then the equations

attached to

$$a, b_1, \dots, b_r, b_*$$

are inconsistent. These at most $n + 2$ slopes give a witness.

Sufficiency is monotone under adjoining slopes, since centre sets shrink. Thus (6.6) is equivalent to inclusion-minimality. For each deletion with a nonempty centre set, Proposition 6.2 supplies a nontrivial all-slope counterexample for the remaining test. These are counterexamples to the weakened assertion itself, not merely failures of the particular proof.

To prove sharpness, put

$$(6.7) \quad S_n = \{0, e_1, \dots, e_n, 2(e_1 + \dots + e_n)\}.$$

The centre equations for $0, e_1, \dots, e_n$ force $\beta_i = 1/2$. The last point $v = 2 \sum_i e_i$ would require $v \cdot \beta = |v|^2/2$, or $n = 2n$, a contradiction. Hence S_n is rigid.

Every $n + 1$ point subset is affinely independent. This is immediate if the last point is removed. If e_j is removed, the vectors e_i for $i \neq j$ together with v are linearly independent, by their j -th components. If 0 is removed, an affine dependence

$$\sum_i c_i e_i + c_v v = 0, \quad \sum_i c_i + c_v = 0$$

gives $c_i = -2c_v$ for all i , then $(1 - 2n)c_v = 0$, so all coefficients vanish. Each such subset has a unique circumcentre obtained by a nonsingular rational linear system; it is rational. All proper subsets therefore have a rational centre. This proves inclusion-minimality and sharpness. $\square$

The proof also gives a finite test procedure: form the centre equations, decide their real consistency, and, if they are consistent, decide whether their affine solution space contains a rational vector. For rational input slopes this is ordinary exact Gaussian elimination. For arbitrary real input, the rational-solvability question is part of the mathematical condition and is not replaced by a floating-point tolerance.

6.5. *Which assumptions have, and have not, been minimized.* The following is the precise sense in which the assumptions are minimal. Within the class of prescribed common-volume slope tests, (6.1) classifies every sufficient test and every counterexample configuration. Proposition 6.3 proves the smallest cardinalities. Theorem 6.6 classifies deletion-minimal tests and gives a sharp bound on their size. These statements remain true with ordinary all-slope minimizing foliations as background, by the converse examples.

Common compatibility cannot be replaced by each foliation having its own invariant volume. For any finite prescribed S , choose a rational $\beta \notin S$. The rational-shear example has rigid minimizing foliations, hence individual invariant phase volumes, at every slope in S , while $V_u \neq 0$. For a rigid test S , Theorem 6.1

proves that these individual volumes cannot be made common. This also shows why independent relabelling is not a substitute for the hypothesis.

No assertion of optimal differentiability thresholds is made. Nor does this classification imply that every sufficient condition imaginable must contain the common-volume condition. It answers the universal implication in Theorem 6.1, with an exact counterexample for every slope set outside it. Rational identity holonomy alone is a different condition. Its local rigidity theorem is proved in Appendix C; the unrestricted global implication is not asserted here.

7. A spectral–Poisson normal form in arbitrary affine rank

The rational-sphere criterion decides which slope sets can support a non-trivial common-volume family. The same reduction also describes every such family. No affine spanning condition is needed in this section.

Fix $a \in S$, and define the allowed Fourier set

$$(7.1) \quad \mathcal{L}(S) = \left\{ (k, m) \in \mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\}) : (b - a) \cdot k + \frac{|b|^2 - |a|^2}{2} m = 0 \quad (b \in S) \right\}.$$

Changing the anchor does not change this set: subtracting the equations for two choices gives exactly the equation for their difference. We exclude $m = 0$ because the mean-removing normalization excludes all purely horizontal modes. In particular,

$$(7.2) \quad \mathcal{L}(S) \neq \emptyset \iff \mathcal{C}_{\mathbb{Q}}(S) \neq \emptyset.$$

Indeed, an allowed (k, m) gives the centre $-k/m$, and a rational centre can be written that way.

THEOREM 7.1 (Spectral–Poisson reconstruction). *Let $S \subset \mathbb{R}^n$ contain at least two distinct points. Choose a real function $\chi \in C^\infty(\mathbb{T}^n \times \mathbb{T})$ such that*

$$(7.3) \quad \int_0^1 \chi(x, s) ds = 0, \quad q_*(x, s) := 1 + \chi_s(x, s) > 0, \quad \text{supp } \widehat{\chi} \subset \mathcal{L}(S).$$

For $b \in S$ set

$$(7.4) \quad v_b(x, s) = \frac{b - \nabla_x \chi(x, s)}{q_*(x, s)}, \quad J_b(x) = \int_0^1 v_b(x, s) ds.$$

Let $f \in C^\infty(\mathbb{T}^n)$ be the unique solution of

$$(7.5) \quad \Delta f = \text{div}_x J_a, \quad \int_{\mathbb{T}^n} f dx = 0.$$

Then, with $s = u + f(x)$,

$$(7.6) \quad \tau_b(x, u) = s + \chi(x, s) - b \cdot x,$$

$$(7.7) \quad q(x, u) = q_*(x, s), \quad p_b(x, u) = v_b(x, s) - \nabla f(x)$$

define rigid solution foliations for one smooth periodic source Q with zero vertical mean. The source is

$$(7.8) \quad Q(x, u) = \left[\operatorname{div}_x v_a + \partial_s \left(\frac{1}{2} |v_a|^2 \right) \right] (x, s) - \Delta f(x).$$

Here the derivatives in the bracket keep s fixed when differentiating in x . Every potential

$$(7.9) \quad V(x, u) = -\partial_u^{-1} Q(x, u) + A(x), \quad A \in C^\infty(\mathbb{T}^n),$$

where $\partial_u^{-1} Q$ is the zero-vertical-mean periodic primitive, has these foliations as class-A minimizers. They preserve the common probability volume $q \, dm$.

Conversely, every smooth common-volume family at S is obtained by this construction, after choosing common additive phase constants so that $\int f = 0$. The Poisson equation is independent of the choice of $a \in S$. The family is nontrivial, in the sense $V_u \not\equiv 0$, exactly when $\chi \neq 0$.

Proof. We prove reconstruction first, and then completeness of the normal form.

The transport equations. The Fourier support condition in (7.3) is equivalent to

$$(7.10) \quad h \cdot \nabla_x \chi + \delta \chi_s = 0, \quad h = b - a, \quad \delta = (|b|^2 - |a|^2)/2,$$

for every $b \in S$. This follows coefficient by coefficient from the smooth Fourier series. In particular,

$$h \cdot \nabla_x q_* + \delta (q_*)_s = 0.$$

Also $v_b - v_a = h/q_*$. The identity $h \cdot a = \delta - |h|^2/2$ and (7.10) give

$$(7.11) \quad h \cdot v_a = \frac{h \cdot a + \delta \chi_s}{q_*} = \delta - \frac{|h|^2}{2q_*}.$$

Thus the residuals in the (x, s) variables agree:

$$(7.12) \quad \begin{aligned} \operatorname{div}_x (v_b - v_a) + \partial_s \left(\frac{|v_b|^2 - |v_a|^2}{2} \right) &= -\frac{h \cdot \nabla_x q_*}{q_*^2} + \partial_s \left(\frac{h \cdot v_a}{q_*} + \frac{|h|^2}{2q_*^2} \right) \\ &= \frac{\delta (q_*)_s}{q_*^2} + \partial_s \left(\frac{\delta}{q_*} \right) = 0. \end{aligned}$$

The Poisson equation. The vector field J_a is smooth and periodic, and its divergence has mean zero on $\mathbb{T}^n$. There is a unique smooth solution of (7.5); explicitly, its nonzero Fourier coefficients are

$$\widehat{f}(k) = -\frac{2\pi i k \cdot \widehat{J}_a(k)}{4\pi^2 |k|^2} \quad (k \neq 0), \quad \widehat{f}(0) = 0.$$

This is a linear equation on the fixed base torus. To see independence of the anchor, put $M_*(x) = \int_0^1 q_*^{-1}(x, s) ds$. Then

$$J_b - J_a = hM_*, \quad h \cdot \nabla_x M_* = \delta \int_0^1 \frac{(q_*)_s}{q_*^2} ds = 0.$$

Therefore $\operatorname{div}_x J_b = \operatorname{div}_x J_a$.

Returning to physical coordinates. Since $s = u + f(x)$, direct differentiation of (7.6) gives

$$(\tau_b)_u = q_*, \quad \nabla_x \tau_b = q_* \nabla f + \nabla_x \chi - b.$$

This proves (7.7). All corrections are periodic, and $\partial_u \tau_b > 0$. Hence each phase defines a global graph foliation with slope b and rigid deck law. In particular its tangent distribution is integrable and its invariant-volume equations follow from closedness of $d\tau_b$.

The residual under this shear is worth recording in full. For $p_b(x, u) = v_b(x, u + f(x)) - \nabla f(x)$,

$$\begin{aligned} \operatorname{div}_x p_b &= \operatorname{div}_x^s v_b + (v_b)_s \cdot \nabla f - \Delta f, \\ \partial_u \left(\frac{1}{2} |p_b|^2 \right) &= (v_b - \nabla f) \cdot (v_b)_s. \end{aligned}$$

Adding cancels the two terms containing $\nabla f \cdot (v_b)_s$. Equation (7.12) now proves that all physical residuals equal the Q in (7.8). Its vertical mean is

$$\int_0^1 Q(x, u) du = \operatorname{div}_x J_a - \Delta f = 0.$$

Thus (7.9) is well defined, smooth and periodic, and $Q = -V_u$. Lemma 4.5 proves class-A minimality against arbitrary fixed-trace competitors. Finally $\int_0^1 q du = 1$, since it is the translated fibre integral of $1 + \chi_s$.

Every common-volume family has this form. Starting with such a family, phase locking gives (5.5). Choose its common additive constant so that $\int_M \psi dm = 0$, and define f, χ by (5.6). Lemma 5.4 gives (7.3) and its allowed Fourier support. The chain rule gives (7.6)–(7.7). The residual computation just performed and the physical zero-mean condition for Q force (7.5). The potential is consequently (7.9) up to its arbitrary u -independent part.

Nontriviality. If $\chi = 0$, then $q_* = 1$, $J_a = a$ and $f = 0$, so $Q = 0$. Conversely, if $Q = 0$, every leaf is harmonic and has bounded deviation from its affine slope. The bounded harmonic difference is constant: for example the interior gradient estimate on balls, followed by letting the radius tend to infinity, makes its gradient zero. Thus $p_b = b$ for all $b \in S$. For a distinct pair, (5.1) gives $h = h/q$, so $q = 1$. Then $\chi_s = 0$, and its zero vertical mean forces $\chi = 0$. $\square$

7.1. *A family which is not a single travelling profile.* The full-span classification below will involve only one travelling variable. That conclusion would be false without full span, even though the rational-sphere criterion remains exact.

Take $n = 2$, $S = \{0, 2e_1\}$, and
(7.13)

$$\chi(x, s) = \varepsilon a(x_2) \sin(2\pi(s - x_1)), \quad a(x_2) = \cos(2\pi x_2), \quad 0 < |\varepsilon| < \frac{1}{2\pi}.$$

Then $(\partial_{x_1} + \partial_s)\chi = 0$ and $q_* = 1 + \chi_s > 0$, exactly as required by (7.10). With $\theta = 2\pi(s - x_1)$,

$$(v_0)_1 = 1 - q_*^{-1}, \quad (v_0)_2 = -\frac{\varepsilon a'(x_2) \sin \theta}{1 + 2\pi \varepsilon a(x_2) \cos \theta}.$$

The vertical mean of the second component is zero by the substitution $\theta \mapsto 2\pi - \theta$. The mean of the first depends only on x_2 . Hence $\operatorname{div}_x J_0 = 0$ and $f = 0$. Formula (7.8) constructs the potential directly, and Theorem 7.1 makes it nontrivial.

The density has nonzero Fourier modes on two different lines, corresponding to $(-1, 1, 1)$ and $(-1, -1, 1)$ in (x_1, x_2, s) . It is therefore not a function of a single integer linear form. Its two foliations have common volume, but the full-span single-profile conclusion does not apply. This example illustrates the additional families retained by the arbitrary-rank normal form. No claim of ordinary foliations at all other slopes is made for this general reconstruction; the all-slope converse needed for Theorem 6.1 is supplied separately in Proposition 6.2.

7.2. *Full-span classification and the exceptional rational centre.* We now impose full affine span and recover an explicit classification without a Poisson correction. This is a consequence of the arbitrary-rank normal form, not an additional rigidity assumption in Theorem 6.1.

THEOREM 7.2 (Full-span classification). *Suppose a common-volume collection contains $n+1$ affinely independent slopes. Let β and $R > 0$ be the centre and radius of their circum-sphere. If $V_u \not\equiv 0$, then $\beta \in \mathbb{Q}^n$. Write $\beta = -k/\ell$, where $\ell > 0$ and $(k_1, \dots, k_n, \ell)$ is primitive. There are smooth functions $r : \mathbb{T} \rightarrow (0, \infty)$ and $A : \mathbb{T}^n \rightarrow \mathbb{R}$ such that*

$$(7.14) \quad \int_0^1 r(z) dz = 1, \quad r \text{ is nonconstant},$$

and

$$(7.15) \quad q(x, u) = r(\ell u + k \cdot x), \quad V(x, u) = A(x) - \frac{R^2}{2r(\ell u + k \cdot x)^2}.$$

Every slope in the common-volume collection lies on $|\alpha - \beta| = R$, and

$$(7.16) \quad p_\alpha(x, u) = \beta + \frac{\alpha - \beta}{q(x, u)}.$$

Conversely, every rational β , every $R > 0$, and every r satisfying (7.14) give calibrated solution foliations at all slopes on this sphere, preserving the same volume. The function $A(x)$ is arbitrary.

Proof. Phase locking and the normal form apply. Choose an anchor a among the affinely independent slopes. Since their differences span $\mathbb{R}^n$, the transport equations imply

$$(7.17) \quad \nabla_x \chi + \beta \chi_s = 0.$$

Indeed, the equations are $h \cdot \nabla_x \chi + (h \cdot \beta) \chi_s = 0$ on a basis of h 's. The nontriviality clause of Theorem 7.1 gives $\chi \neq 0$. A nonzero Fourier mode (k', m) therefore satisfies $k' + m\beta = 0$ and $m \neq 0$, which proves that β is rational.

For primitive (k, ℓ) as stated, the integer solutions of $k' + m\beta = 0$ are exactly $(k', m) = j(k, \ell)$ with $j \in \mathbb{Z}$. To verify this, the equation is $\ell k' = mk$, and a Bezout combination of $k_1, \dots, k_n, \ell$ shows that ℓ divides m . Hence Fourier uniqueness gives

$$\chi(x, s) = F(\ell s + k \cdot x)$$

for a smooth periodic F . Put $r = 1 + \ell F'$, so $r > 0$ and $\int r = 1$. Equation (7.17) gives

$$v_\alpha = \beta + \frac{\alpha - \beta}{r(\ell s + k \cdot x)}.$$

Its vertical average J_α is constant in x , because integration covers exactly ℓ periods of r^{-1} . Thus (7.5) gives $f = 0$.

Theorem 5.5 makes all slopes cospherical; uniqueness of the centre of the affinely independent subset identifies its centre as this β . For $c = \alpha - \beta$, $|c| = R$, we have $p_\alpha = \beta + c/q$ and $\nabla_x q = -\beta q_u$. Direct calculation gives

$$\begin{aligned} \operatorname{div}_x p_\alpha &= \frac{c \cdot \beta}{q^2} q_u, \\ \partial_u \left(\frac{1}{2} |p_\alpha|^2 \right) &= -\frac{\beta \cdot c}{q^2} q_u - \frac{|c|^2}{q^3} q_u. \end{aligned}$$

Therefore $Q = -R^2 q_u / q^3 = \partial_u (R^2 / (2q^2))$. Integration of $Q = -V_u$ proves (7.15). The function r is nonconstant because the constant case makes $Q = 0$.

For the converse, choose H with $H' = r$ and $H(z+1) = H(z) + 1$. Define

$$(7.18) \quad \tau_\alpha(x, u) = \frac{H(\ell u + k \cdot x)}{\ell} - (\alpha - \beta) \cdot x.$$

Then $\tau_\alpha = u - \alpha \cdot x +$ a periodic correction, $\partial_u \tau_\alpha = q > 0$, and its field is (7.16). The residual just calculated equals $-V_u$, and mixed phase derivatives give common invariance. Lemma 4.5 proves class-A minimality against all-dimensional competitors. $\square$

7.2.1. *The centre cannot carry a closed minimizing foliation.*

LEMMA 7.3 (Resonant obstruction for a travelling potential). *Let $\beta = -k/\ell \in \mathbb{Q}^n$, $\ell > 0$, and*

$$V(x, u) = A(x) + F(\ell u + k \cdot x)$$

with F smooth periodic and nonconstant. There is no class-A minimizing foliation at slope β whose leaves close with their prescribed increments on a full-rank finite cover.

Proof. For a leaf u , put $y = \ell u + k \cdot x$. The action identity is

$$(7.19) \quad \begin{aligned} \mathcal{A}_V[u; \Omega] = & \frac{1}{\ell^2} \int_{\Omega} \left(\frac{1}{2} |\nabla y|^2 - \ell^2 F(y) \right) dx - \frac{1}{\ell^2} \int_{\Omega} k \cdot \nabla y dx \\ & + \frac{|k|^2}{2\ell^2} |\Omega| - \int_{\Omega} A(x) dx. \end{aligned}$$

The last three terms are fixed for fixed-trace competitors. At slope β , a closed leaf makes y periodic on the corresponding full-rank lattice. Adding $\ell^2 \max F$ to the transformed density leaves minimality unchanged and gives the nonnegative potential

$$W(y) = \ell^2 (\max F - F(y)).$$

The periodic replacement lemma in Section 2 forces y to be constant at a maximum of F . Those values form a proper closed subset of the vertical circle because F is nonconstant. Such constants cannot cover all values of y at a fixed x , whereas a foliation must. This excludes any alternative closed minimizing foliation, not merely the phase choice in (7.18). $\square$

COROLLARY 7.4 (Finite centre test). *Suppose $n + 1$ affinely independent solution foliations share a positive invariant volume, and let β be their circumcentre. Then $V_u = 0$ if either $\beta \notin \mathbb{Q}^n$, or $\beta \in \mathbb{Q}^n$ and there is a closed minimizing foliation at β . The centre foliation need not preserve the common volume.*

Proof. If $V_u \neq 0$, Theorem 7.2 makes β rational and gives a nonconstant travelling potential centred at β . Lemma 7.3 then excludes the second possibility as well. $\square$

Consequently, rational closed holonomy at every rational slope, together with common volume at $n + 1$ affinely independent slopes, forces $V_u = 0$. This combines two distinct types of information. It does not derive common volume from holonomy, and it is not needed for the smaller arithmetic tests in Section 6. AI use statement. ChatGPT (including GPT-5.5, GPT-5.6, and GPT-6) was used to prepare and edit the L^AT_EX source (and turn handwritten notes into L^AT_EX), assist with literature searches, critically assess proposed approaches,

help develop and refine mathematical statements and proofs, and review the manuscript.

In particular, ChatGPT contributed to the development and refinement of the all-dimensional counterexample in Section 2, including the square-completion calibration establishing minimality against competitors depending on all base variables. It also assisted in distinguishing ordinary minimizing foliations from foliations with closed rational holonomy, and in developing and checking the rational-shear construction that places the intrinsic holonomy obstruction at an arbitrary prescribed rational slope in Section 3.

ChatGPT contributed to the phase formulation in Section 4 and to the development and refinement of the main common-volume rigidity argument. This included the passage from a common invariant volume to closed phase forms, the phase-locking and mean-removing shear arguments in Section 5, including Lemmas 5.1 and 5.4, and the reduction to constant-coefficient transport equations. ChatGPT also assisted with the Fourier argument identifying the rational-centred sphere obstruction and with the exact rigidity criterion and optimal finite-slope tests in Section 6.

ChatGPT contributed to the development and refinement of the spectral-Poisson reconstruction in Section 7, including the treatment of arbitrary affine rank, the full-span classification, and the obstruction at the exceptional rational centre. These contributions concerned both the compatibility calculations and the reconstruction of families realizing the exceptional slope configurations.

Assistance with the appendices included the formulation, critical review, and exposition of the rational-cell, quantitative phase-escape, and local identity-holonomy arguments in Appendices A–C, and of the extensions to anisotropic and uniformly convex kinetic energies, nonlinear elliptic selection, analytic compatibility, and finite-jet compatibility in Appendices D–H. ChatGPT also assisted with checking supporting identities and examples and with clarifying the hypotheses needed for the auxiliary arguments.

During the research process, ChatGPT was also used to examine earlier proposed proof strategies, identify unjustified steps, and test proposed shortcuts concerning slope dependence and continuation of foliations. These investigations helped clarify the distinction between the common-volume rigidity theorem proved here and the unresolved global implication from rational identity holonomy alone. Unresolved steps from earlier exploratory arguments are not used to establish the main theorem.

The author takes full responsibility for the correctness and final content of the work, including all mathematical statements, proofs, citations, and references, and all material developed with the assistance of generative AI.

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Appendix A. Rational cells, order, and exact barriers

Fix $\alpha \in \mathbb{Q}^n$ and an integer $L \geq 1$ with $L\alpha \in \mathbb{Z}^n$. We may work on $C = \mathbb{R}^n / L\mathbb{Z}^n$. A graph of slope α is written

$$(A.1) \quad v(x) = \alpha \cdot x + w(x), \quad w \text{ is } L\mathbb{Z}^n\text{-periodic.}$$

Its mean height is $\langle w \rangle_C$. The integrand of its cell energy is periodic. More general full-rank finite-index sublattices of $\mathbb{Z}^n$ are handled identically.

LEMMA A.1 (Existence and stability of cell minima). *The action has a global minimum in the class (A.1). Every minimizer is smooth and its periodic Jacobi operator*

$$(A.2) \quad \mathcal{L}_v = -\Delta + Q_u(x, v) = -\Delta - V_{uu}(x, v)$$

is nonnegative. A minimum also exists with the additional constraint $\langle w \rangle_C = c$, for every $c \in \mathbb{R}$.

Proof. The cross term $\langle \alpha \cdot \nabla w \rangle_C$ is zero. Boundedness of V therefore bounds the gradient of a minimizing sequence. In the unconstrained case, subtract vertical integers so that its mean lies in $[0, 1]$. Poincaré's inequality then bounds its H^1 norm. Weak H^1 compactness and strong L^2 compactness give a limit; a subsequence converges almost everywhere, so bounded convergence treats the potential term. The gradient norm is weakly lower semicontinuous. This proves existence, with the same argument at prescribed mean. The unconstrained first variation gives the equation in (2.1); boundedness of its right side gives $W^{2,p}$ regularity for every finite p , followed by smooth elliptic bootstrapping. Its second variation is the quadratic form of (A.2) and is nonnegative. For a constrained minimum a scalar multiplier appears; it must not be silently set to zero. $\square$

LEMMA A.2 (Order of global cell minima). *Any two global cell minimizers of the same slope are either equal or strictly ordered. In particular, there is at most one global cell minimizer of each mean height.*

Proof. For two minimizers v, w , their pointwise minimum and maximum belong to the same affine class and satisfy

$$E_V[\min(v, w)] + E_V[\max(v, w)] = E_V[v] + E_V[w].$$

This is the Sobolev min-max identity: almost everywhere the two values and the two gradients have merely been interchanged. Both new functions must be global minima, hence smooth solutions. The nonnegative difference $\max(v, w) - v$ solves a linear equation $\Delta d = A(x)d$ with bounded coefficient. Harnack's inequality implies that it is identically zero or everywhere positive. In the latter case $w > v$ everywhere. In the former case $w \leq v$, and the same argument applied to $v - w$ gives equality or strict inequality. Equal mean heights rule out either strict ordering. $\square$

LEMMA A.3 (Sliding among solutions). *If a closed ordered solution foliation of slope α exists on C , every other solution with the same increments is one of its leaves.*

Proof. Let τ be its lifted inverse phase and v the other solution. The function $x \mapsto \tau(x, v(x))$ is periodic on C , because the affine increments cancel.

At its maximum t_* , the difference $U_{t_*} - v$ is nonnegative and has a zero. It satisfies a bounded-coefficient scalar linear elliptic equation, so Harnack gives equality. The foliation contains exactly one leaf of each mean height. $\square$

LEMMA A.4 (Calibration and natural closure). *A smooth closed solution foliation calibrates every leaf against arbitrary all-dimensional fixed-trace competitors and against all same-slope periodic competitors. A closed solution foliation on any finite rational cover descends, as a family, to the original torus and has identity adjusted holonomy on the natural cover.*

Proof. Let $p(x, u)$ be its slope field on the cover. The equation gives

$$\operatorname{div}_x p + \partial_u \left(\frac{1}{2} |p|^2 + V \right) = 0.$$

Write $dx = dx_1 \wedge \cdots \wedge dx_n$ and $dx_i^\# = \iota_{\partial_{x_i}} dx$. Then

$$\omega_p = \sum_i p_i du \wedge dx_i^\# - \left(\frac{1}{2} |p|^2 + V \right) dx$$

is closed, and every graph satisfies

$$(A.3) \quad \left(\frac{1}{2} |\nabla v|^2 - V(x, v) \right) dx = v^* \omega_p + \frac{1}{2} |\nabla v - p(x, v)|^2 dx.$$

The integral of the first term agrees for fixed-trace graphs, by Stokes and approximation, and for same-slope closed graphs, by homology. Equality holds for each leaf. This proves both minimizing assertions.

Normalize the leaves by $\langle U_c - \alpha \cdot x \rangle_C = c$. Vertical translation by one and horizontal translation by any $m \in \mathbb{Z}^n$ give solutions on the same cover. Lemma A.3 and the means imply

$$(A.4) \quad U_{c+1}(x) = U_c(x) + 1, \quad U_c(x + m) = U_{c+\alpha \cdot m}(x).$$

Thus p is periodic on the original torus. When $\alpha \cdot m$ is integral, (A.4) makes each leaf close with increment $\alpha \cdot m$, which is precisely natural rational closure. $\square$

For completeness, the reverse cell comparison is also valid: a class-A minimizer with periodic corrector is a global cell minimizer. A lower-energy smooth cell competitor can be repeated in the interior of a cube containing R^n cells and joined to the original graph in a collar of fixed width. The energy gain is of order R^n , whereas the collar cost is $O(R^{n-1})$, since the difference of the two periodic correctors and their derivatives is bounded. Large R contradicts class-A minimality. Approximation treats Sobolev competitors.

A.1. *An exact mean-gap criterion.* Define

$$(A.5) \quad \begin{aligned} e_\alpha(V; c) &= \inf_{\substack{w \in H^1(C) \text{ periodic} \\ \langle w \rangle_C = c}} \left\langle \frac{1}{2} |\alpha + \nabla w|^2 - V(x, \alpha \cdot x + w) \right\rangle_C, \\ B_\alpha(V) &= \max_{c \in [0,1]} e_\alpha(V; c) - \min_{c \in [0,1]} e_\alpha(V; c). \end{aligned}$$

The dependence on the fixed cover is suppressed. Vanishing, unlike the numerical value, is independent of this choice.

THEOREM A.5 (Mean-gap characterization and robustness). *The following are equivalent: $B_\alpha(V) = 0$; a closed minimizing foliation of slope α exists; and a smooth closed ordered solution foliation of slope α exists. Furthermore,*

$$(A.6) \quad |B_\alpha(V) - B_\alpha(W)| \leq 2\|V - W\|_\infty.$$

Proof. Vertical integer translation makes e_α one-periodic in c . Translating any competitor by a constant h gives

$$|e_\alpha(V; c + h) - e_\alpha(V; c)| \leq \|V_u\|_\infty |h|,$$

so its maximum and minimum exist. If $B_\alpha = 0$, the constrained minima from Lemma A.1 are all global cell minima. Lemma A.2 gives unique, strictly ordered minima U_c at every mean c . Compactness at bounded mean and uniqueness give continuous dependence in spatial C^2 on c . Here the right sides are bounded uniformly; elliptic estimates and bootstrapping upgrade subsequential convergence to C^2 before uniqueness identifies the limit. The family fills every vertical fibre because $U_{c+1} = U_c + 1$ and is strictly ordered.

We spell out transverse regularity. The positive differences

$$z_h = \frac{U_{c+h} - U_c}{h}, \quad \langle z_h \rangle_C = 1,$$

solve $\Delta z_h = A_h z_h$ with uniformly bounded coefficients. On the fixed compact cell, Harnack gives uniform positive upper and lower bounds. Elliptic estimates give compactness, and every limit is a positive Jacobi field of mean one. The nonnegative Jacobi operator has a simple ground eigenspace, so the limit is unique. Thus $\partial_c U_c = \phi_c > 0$ exists. For higher regularity use the bordered equation

$$-\Delta w + Q(x, \alpha \cdot x + c + w) - \mu = 0, \quad \langle w \rangle_C = 0.$$

At a leaf its linearization is $(\eta, \nu) \mapsto \mathcal{L}_{U_c} \eta - \nu$ on mean-zero η . Pairing with the positive normalized kernel ϕ_c fixes ν ; the Fredholm alternative solves for η , with its remaining kernel multiple fixed by zero mean. The inverse is bounded by elliptic estimates. The smooth implicit-function theorem applies, and uniqueness captures the existing leaves on this branch with $\mu = 0$. Hence the foliation is smooth. Lemma A.4 supplies class-A minimality and natural closure.

Conversely, a closed minimizing foliation consists of global cell minima by the comparison following Lemma A.4. Its mean heights cover $\mathbb{R}$, so e_α is constant. A smooth solution foliation has the same property by calibration. Finally, for each competitor the normalized energies for V and W differ by at most $\|V - W\|_\infty$. Taking infima at fixed c gives the same bound for the two e_α functions. Their oscillations differ by at most twice this quantity, proving (A.6). $\square$

Remark A.6. The prescribed mean is essential. Minimizing over an unrestricted vertical constant would make the displayed energy independent of a nominal phase parameter for every potential. Nor is point evaluation used as a Sobolev constraint in higher dimensions. The mean is a continuous functional on the compact-cell H^1 space.

A.2. *Exact pinning by a phase perturbation.* Suppose the rational foliation at α exists, and let $\tau(x, u)$ be its mean-normalized inverse phase. Choose $\ell \in \mathbb{N}$ with $\ell\alpha \in \mathbb{Z}^n$ and a smooth nonconstant one-periodic function f . The function

$$(A.7) \quad W(x, u) = f(\ell\tau(x, u)), \quad V_s = V + sW$$

is genuinely periodic on the original physical torus: all deck increments of $\ell\tau$ are integers.

THEOREM A.7 (An entire nonzero parameter line is pinned). *For the deformation (A.7), the rational slope α has a closed minimizing foliation exactly when $s = 0$. If E_* is the original normalized minimum, then*

$$(A.8) \quad \min E_{V_s} = \begin{cases} E_* - s \max f, & s > 0, \\ E_* - s \min f, & s < 0. \end{cases}$$

For $s > 0$, the entire set of periodic cell minimizers consists of the original leaves for which $f(\ell\tau) = \max f$. For $s < 0$ it consists of the corresponding minimizing phases of f .

Proof. By (A.3), for every same-slope periodic competitor,

$$(A.9) \quad E_{V_s}[v] = E_* + \frac{1}{2} \langle |\nabla v - p(x, v)|^2 \rangle_C - s \langle f(\ell\tau(x, v)) \rangle_C.$$

For $s > 0$, both terms in

$$E_{V_s}[v] - (E_* - s \max f) = \frac{1}{2} \langle |\nabla v - p(x, v)|^2 \rangle_C + s \langle \max f - f(\ell\tau(x, v)) \rangle_C$$

are nonnegative. An original leaf at a maximum of f attains equality. Equality for any other minimum forces $\nabla v = p(x, v)$ everywhere, and then

$$\nabla_x[\tau(x, v(x))] = \tau_x + \tau_u \nabla v = 0.$$

Thus v is an original leaf, whose phase must maximize f . Because f is nonconstant, those phases do not fill the vertical circle. Theorem A.5 rules out a closed foliation. The negative case is identical with $\min f$. The argument also works on every larger finite cover; it is not merely a failure on one chosen cell. $\square$

COROLLARY A.8 (Generic absence of every rational closed foliation). *In the smooth topology, the potentials with no closed minimizing foliation at any rational slope form a residual set. For each fixed rational slope the absence set is open and dense. The locus satisfying $\text{RH}_{\mathbb{Q}}$ is closed relative to smooth potentials in the uniform topology and is nowhere dense in the smooth topology.*

Proof. For a fixed slope, the absence set is $\{B_{\alpha} > 0\}$, open by (A.6). At every point of its complement, Theorem A.7 supplies arbitrarily small smooth perturbations in that set, proving density. Intersect over the countable set $\mathbb{Q}^n$ and apply the Baire theorem to $C^{\infty}(M)$. The all-rational locus is the intersection of the closed zero sets of B_{α} and is contained in any one of these nowhere dense zero sets. $\square$

This is a generic statement, not a classification of every potential. The existence of a nowhere dense exceptional locus does not show that it consists only of the u -independent potentials. In particular, this argument does not remove an isolated nonzero all-rational coupling allowed by the analytic dichotomy in Appendix K.

Appendix B. Quantitative phase escape without common volume

The common-volume theorem compares the densities of distinct foliations exactly. There is another useful result when those densities need not agree. It comes from one admissible phase at a time and has its own sharpness statement.

For an admissible solution phase $\tau_{\alpha} = u - \alpha \cdot x + \psi_{\alpha}$, put

$$q_{\alpha} = (\tau_{\alpha})_u, \quad p_{\alpha} = -\frac{\nabla_x \tau_{\alpha}}{q_{\alpha}}, \quad \mu_{\alpha} = q_{\alpha} dm,$$

and define

$$(B.1) \quad D(\tau_{\alpha}) = \int_M V(q_{\alpha} - 1) dm.$$

This density depends on the admissible phase parametrization. The inequalities below hold for every such choice; no canonical rational relabelling is presumed.

B.1. The exact affine-phase defect.

THEOREM B.1 (Symmetrized affine defect). *Every admissible solution phase satisfies*

$$(B.2) \quad D(\tau_{\alpha}) = \frac{1}{2} \int_M (1 + q_{\alpha}) |p_{\alpha} - \alpha|^2 dm \geq 0.$$

Equality for one phase forces $V_u = 0$. Conversely, if $V_u = 0$, the defect of every admissible phase is zero.

Proof. For $\eta > 0$ define the perspective integrand

$$F_z(\xi, \eta) = \frac{|\xi|^2}{2\eta} - V(z)\eta, \quad z = (x, u).$$

The actual phase form is $\theta = (-qp, q)$, while the affine form is $\theta^0 = (-\alpha, 1)$. Direct expansion gives

$$(B.3) \quad F_z(\theta^0) - F_z(\theta) - DF_z(\theta) \cdot (\theta^0 - \theta) = \frac{1}{2}|p - \alpha|^2.$$

Here

$$DF_z(\theta) = \left(-p, -\frac{1}{2}|p|^2 - V \right)$$

is divergence-free by the residual equation, and $\theta^0 - \theta = -d\psi$ is exact and periodic. Hence the integrated linear term vanishes. With

$$\mathcal{J}_V(\tau) = \int_M q \left(\frac{1}{2}|p|^2 - V \right) dm,$$

we obtain

$$(B.4) \quad \mathcal{J}_V(\tau^0) - \mathcal{J}_V(\tau) = \frac{1}{2} \int_M |p - \alpha|^2 dm.$$

Evaluating the actions also gives

$$\mathcal{J}_V(\tau^0) - \mathcal{J}_V(\tau) = \frac{1}{2}|\alpha|^2 - \frac{1}{2} \int_M q|p|^2 dm + D(\tau).$$

The cohomology identities $\int q = 1$ and $\int qp = \alpha$ imply

$$\int_M q|p - \alpha|^2 dm = \int_M q|p|^2 dm - |\alpha|^2.$$

Combining these equations proves (B.2). If it vanishes, $p = \alpha$ and the residual equation gives $Q = 0$. If $Q = 0$, each leaf is harmonic with bounded affine deviation, so is affine as in Theorem 7.1; then the right side vanishes. $\square$

There is also a direct proof which makes the physical source visible. Since $\nabla_x \psi = \alpha - qp$ and $\psi_u = q - 1$, integration of $Q\psi$ gives

$$\begin{aligned} \int_M Q\psi dm &= - \int_M p \cdot \nabla_x \psi dm - \frac{1}{2} \int_M |p|^2 \psi_u dm \\ &= -\alpha \cdot \int_M p dm + \frac{1}{2} \int_M (1 + q)|p|^2 dm \\ &= \frac{1}{2} \int_M (1 + q)|p - \alpha|^2 dm. \end{aligned}$$

The last equality uses the same cohomology identities. On the other hand, $\int Q\psi = -\int V_u\psi = \int V(q-1)$, yielding the theorem again. The perspective proof will extend to nonquadratic kinetic energies in Appendix E.

B.2. *A single Fourier coefficient gives a uniform estimate.*

THEOREM B.2 (Inverse-square lower bound). *Assume $Q \not\equiv 0$. Choose $(k, \ell) \in \mathbb{Z}^n \times \mathbb{Z}$ with*

$$\ell \neq 0, \quad c = |\widehat{Q}(k, \ell)| > 0.$$

Set

$$(B.5) \quad A = 2\pi\sqrt{2(|k|^2 + \ell^2)}, \quad B = 2\pi|\ell|, \quad c_V = \left(\frac{2c}{A + \sqrt{A^2 + 4Bc}} \right)^2.$$

Then $c_V > 0$ and every admissible solution phase satisfies

$$(B.6) \quad D(\tau_\alpha) \geq \frac{c_V}{1 + |\alpha|^2}.$$

Proof. The coefficients with $\ell = 0$ vanish because Q has zero vertical mean. Thus such a nonzero coefficient exists. Put $r = p_\alpha - \alpha$. The residual equation is

$$Q = \operatorname{div}_x r + \partial_u \left(\alpha \cdot r + \frac{1}{2}|r|^2 \right).$$

Testing against $e^{-2\pi i(k \cdot x + \ell u)}$ and integrating by parts yields

$$(B.7) \quad c \leq 2\pi|k + \ell\alpha| \|r\|_{L^1} + \pi|\ell| \|r\|_{L^2}^2.$$

By Theorem B.1, $\|r\|_2^2 \leq 2D$ and $\|r\|_1 \leq \sqrt{2D}$, since $m(M) = 1$. Also

$$|k + \ell\alpha|^2 \leq (|k|^2 + \ell^2)(1 + |\alpha|^2).$$

For $X = (1 + |\alpha|^2)D$, inequality (B.7) therefore gives

$$c \leq A\sqrt{X} + BX.$$

The right side is increasing for $X \geq 0$. Its positive quadratic root in $\sqrt{X}$ is $2c/(A + \sqrt{A^2 + 4Bc})$, so $X \geq c_V$. This is (B.6). $\square$

The lower bound does not require compactness of the phase fields, continuity in slope, or a lower bound on q uniform over the collection. It uses only one fixed nonzero Fourier coefficient of the physical source.

B.3. *Positive ensembles and finite-slope visibility.* A finite positive ensemble consists of admissible phases τ_j at slopes α_j and weights $w_j \geq 0$ with $\sum_j w_j = 1$. Its averaged phase measure and slope distribution are

$$\nu = \sum_j w_j q_j \, dm, \quad \sigma = \sum_j w_j \delta_{\alpha_j}.$$

The same definition applies to a probability measure Π on a measurable family of phases, provided $(\tau, z) \mapsto q_\tau(z)$ and the slope map are measurable. In that case

$$\nu = \int q_\tau dm d\Pi(\tau), \quad \sigma = \alpha_\# \Pi.$$

Tonelli's theorem and $\int q_\tau dm = 1$ make ν a probability measure. Since V is bounded, all integrals of V below are finite.

THEOREM B.3 (Sharp positive-escape alternative). *If $V_u \not\equiv 0$, every positive ensemble satisfies*

$$(B.8) \quad \varepsilon := \int_M V d\nu - \int_M V dm \geq c_V \int_{\mathbb{R}^n} \frac{d\sigma(\alpha)}{1 + |\alpha|^2}.$$

In particular, for every $R < \infty$,

$$(B.9) \quad \sigma(\overline{B}_R) \leq \frac{1 + R^2}{c_V} \varepsilon.$$

If a sequence of positive ensembles has $\varepsilon_j \rightarrow 0$ and

$$(B.10) \quad \limsup_{j \rightarrow \infty} \int_{\mathbb{R}^n} \frac{d\sigma_j(\alpha)}{1 + |\alpha|^2} > 0,$$

then $V_u = 0$.

Proof. The left side of (B.8) is the positive average of the defects (B.1). Average (B.6) to obtain the inequality. On $\overline{B}_R$, $(1 + |\alpha|^2)^{-1} \geq (1 + R^2)^{-1}$, proving (B.9). If $V_u \not\equiv 0$, (B.8) contradicts $\varepsilon_j \rightarrow 0$ and (B.10). $\square$

Condition (B.10) is equivalent to there being a finite R such that $\limsup_j \sigma_j(\overline{B}_R) > 0$. One direction follows from the preceding lower bound. For the other, split the integral over $\overline{B}_R$ and its complement:

$$\int \frac{d\sigma_j}{1 + |\alpha|^2} \leq \sigma_j(\overline{B}_R) + \frac{1}{1 + R^2}.$$

If the left side has positive limsup, choose R large enough and take the corresponding subsequence. Thus only a nonvanishing fraction of the slope mass must remain bounded; tightness of the whole sequence is stronger than needed.

COROLLARY B.4 (Exact balance and compact approximation). *Each of the following conditions forces $V_u = 0$:*

- (i) *a positive ensemble satisfies $\int V d\nu = \int V dm$;*
- (ii) *$\nu_j \rightharpoonup m$ and (B.10) holds;*
- (iii) *Haar measure belongs to the weak closed convex hull of phase measures with slopes in one compact set.*

Proof. For (i), Theorem B.1 says that a nonnegative defect has average zero; it vanishes for almost every phase, and any one such phase gives $V_u = 0$. For (ii), weak convergence tests the continuous function V , giving $\varepsilon_j \rightarrow 0$. For (iii), approximate Haar by finite convex combinations in the compact slope set and apply (ii). Weak convergence of probabilities on the compact metric space M is metrizable, so a sequence suffices. $\square$

The exact one-observable balance is already sufficient. Full equality $\nu = m$, full support of the ensemble, and a globally smooth complete integral are not required. Conversely, if $V_u = 0$, affine phases with $q = 1$ give all these properties. This is an equivalence of phase-completeness conditions, but not a derivation of them from separately existing rational foliations.

B.4. *The counterexample saturates both the order and the escape condition.* Return to $V_b = -b^2/2$ and its nonzero-slope phases. Write

$$r = |\alpha|, \quad A(u) = b(u)^2, \quad c = \sqrt{2E(r)},$$

$$r = \left(\int_0^1 (c^2 + A)^{-1/2} du \right)^{-1}.$$

Then

$$(B.11) \quad q_\alpha(u) = \frac{r}{\sqrt{c^2 + A(u)}}, \quad p_\alpha(u) = \frac{\alpha}{r} \sqrt{c^2 + A(u)}.$$

PROPOSITION B.5 (Sharp asymptotics). *As $r \rightarrow \infty$, for every fixed derivative order,*

$$(B.12) \quad q_\alpha = 1 + \frac{\bar{A} - A}{2r^2} + O_{C^\infty}(r^{-4}),$$

$$(B.13) \quad D(\tau_\alpha) = \frac{\text{Var}(A)}{4r^2} + O(r^{-4}).$$

For $b(u) = \sin^2(\pi u)$, the coefficient in (B.13) is $17/512$.

Proof. Let $m_j = \int_0^1 A^j du$. Uniform binomial expansion gives

$$(c^2 + A)^{-1/2} = c^{-1} \left(1 - \frac{A}{2c^2} + \frac{3A^2}{8c^4} + O_{C^\infty}(c^{-6}) \right).$$

Integration and scalar inversion yield

$$r = c + \frac{m_1}{2c} + \left(\frac{m_1^2}{4} - \frac{3m_2}{8} \right) c^{-3} + O(c^{-5}).$$

Multiply this by the first expansion to obtain

$$q_\alpha = 1 + \frac{m_1 - A}{2c^2} + O_{C^\infty}(c^{-4}).$$

Since $r = c + O(c^{-1})$, replacing c^{-2} by r^{-2} changes only the $O(r^{-4})$ term. Finally,

$$D(\tau_\alpha) = -\frac{1}{2} \int_0^1 A(q_\alpha - 1) du = \frac{m_2 - m_1^2}{4r^2} + O(r^{-4}).$$

For $A = \sin^4(\pi u)$, $m_1 = 3/8$ and $m_2 = 35/128$, so $(m_2 - m_1^2)/4 = 17/512$. $\square$

The notation O_{C^∞} means a bound in each fixed C^j norm, with a constant allowed to depend on j . Because A is nonconstant, $\text{Var}(A) > 0$. Thus no universally stronger decay order can replace r^{-2} in the lower-bound theorem. Also $q_\alpha \rightarrow 1$ smoothly at infinite slope while $V_u \neq 0$. A sequence of these single-phase ensembles satisfies even smooth convergence to Haar, but loses all bounded-slope mass. Hence the finite-slope visibility assumption cannot be deleted from Theorem B.3.

B.5. *Finite translations, Hessian squares, and a leafwise defect.* These identities use actual calibrated fields on the physical torus. They require neither common volume between different foliations nor derivatives with respect to slope.

PROPOSITION B.6 (Cross-translation identity). *Let τ_a, τ_b be smooth admissible solution phases for the same potential. For every $y \in M$,*

$$\begin{aligned} & \int_M q_b(z) (V(z) - V(z - y)) dm(z) \\ (B.14) \quad &= \frac{1}{2} \int_M q_b(z) \left(|p_b(z) - p_a(z - y)|^2 \right. \\ & \quad \left. - |p_b(z) - p_a(z)|^2 \right) dm(z). \end{aligned}$$

In particular, taking $a = b$ gives a nonnegative right side.

Proof. Put

$$X_a(z) = \left(-p_a(z), -\frac{1}{2}|p_a(z)|^2 - V(z) \right), \quad \theta_b = d\tau_b = (-q_b p_b, q_b).$$

The residual equation gives $\text{div}_{x,u} X_a = 0$. The pairing

$$I(y) = \int_M X_a(z - y) \cdot \theta_b(z) dm(z)$$

is independent of y . In fact $\theta_b = (-b, 1) + d\psi_b$; the pairing with $d\psi_b$ vanishes by integration by parts, and the pairing with the constant form is translation invariant.

A pointwise computation gives

$$\begin{aligned} q_b(z) \left(\frac{1}{2} |p_b(z)|^2 - V(z) \right) - X_a(z-y) \cdot \theta_b(z) \\ = q_b(z) \left(\frac{1}{2} |p_b(z) - p_a(z-y)|^2 + V(z-y) - V(z) \right). \end{aligned}$$

Subtract its integral at $y = 0$ from that at y . The left side vanishes by constancy of I , proving (B.14). $\square$

COROLLARY B.7 (Ambient Hessian as a Gram matrix). *For every smooth admissible phase,*

$$(B.15) \quad - \int_M q D_{x,u}^2 V \, dm = \int_M q (D_{x,u} p)^T (D_{x,u} p) \, dm \succeq 0.$$

In particular,

$$(B.16) \quad \int_M q Q_u \, dm = \int_M q |p_u|^2 \, dm.$$

Proof. Set $a = b$ in (B.14). Differentiate twice in the translation y at $y = 0$. The field difference vanishes at zero, so only the product of its first derivatives remains. This yields the matrix equality. Its vertical entry is (B.16), since $Q_u = -V_{uu}$. $\square$

The same Hopf identity follows from differentiating the residual equation in u :

$$Q_u = \sum_i D_i(p_{i,u}) + |p_u|^2.$$

Invariance of $q \, dm$ under D_i removes the divergence terms. These are fixed-volume identities, not an argument that averaging over unrelated slopes yields Haar measure.

B.5.1. Localization at an integer-slope leaf.

PROPOSITION B.8 (Leafwise affine defect). *Let $a \in \mathbb{Z}^n$ and let U_a be the inverse of a smooth admissible phase. Put*

$$E_a(t) = \int_{\mathbb{T}^n} |p_a(x, U_a(x, t)) - a|^2 \, dx, \quad D_a^0 = \frac{1}{2} \int_M |p_a - a|^2 \, dm.$$

Then every phase satisfies

$$(B.17) \quad \int_{\mathbb{T}^n} V(x, U_a(x, t)) \, dx - \int_M V \, dm = D_a^0 + \frac{1}{2} E_a(t).$$

If $V_u \neq 0$, the left side is strictly positive on every leaf.

Proof. For any increasing degree-one lift h , the phase $h \circ \tau_a$ is admissible with density $h'(\tau_a)q_a$ and unchanged field p_a . Integrality of a is used here: all

horizontal phase increments are integers, so this relabelling preserves the same cohomology and gives a periodic correction. Theorem B.1 gives

$$\int_M V(h'(\tau_a)q_a - 1) dm = D_a^0 + \frac{1}{2} \int_M h'(\tau_a)q_a |p_a - a|^2 dm.$$

For each fixed x , the substitution $u = U_a(x, t)$ has $q_a du = dt$. The functions in the resulting t -integrals are periodic; integration over any interval of length one agrees. Hence the equality becomes

$$\int_0^1 H(t) \left[\int_{\mathbb{T}^n} V(x, U_a(x, t)) dx - \int_M V dm - \frac{1}{2} E_a(t) \right] dt = D_a^0,$$

where $H = h' > 0$ and $\int_0^1 H = 1$. Every smooth positive probability density H arises in this way. Taking $H = 1 + \varepsilon g$ for arbitrary smooth zero-mean g proves that the bracket is the constant D_a^0 . This is (B.17). If $D_a^0 = 0$, then $p_a = a$ and $Q = 0$; thus it is strictly positive on the nontrivial branch. $\square$

B.6. Leafwise translation and the effective-action defect. Let $a \in \mathbb{Q}^n$ and let $U_a(\cdot, t)$ be a closed leaf on a normalized rational cell C_a . Its leaf action $\mathcal{A}(a)$ is independent of t : all these graphs are homologous and are calibrated by one closed Hilbert form. In this section $\oint_{C_a}$ denotes normalized spatial integration, whereas $\int_M$ is physical Haar integration.

PROPOSITION B.9 (Finite leafwise translation identity). *Let p_b be the field of any other smooth solution foliation for the same potential. With $z_t(x) = (x, U_a(x, t))$ and $y \in M$,*

$$\begin{aligned} & \oint_{C_a} [V(z_t(x)) - V(z_t(x) - y)] dx \\ \text{(B.18)} \quad &= \frac{1}{2} \oint_{C_a} \left[|p_a(z_t(x)) - p_b(z_t(x) - y)|^2 - |p_a(z_t(x)) - p_b(z_t(x))|^2 \right] dx. \end{aligned}$$

In particular the right side is nonnegative when $b = a$.

Proof. Write $y = (c, d)$ and compare $U_a(x, t)$ with $v_y(x) = U_a(x + c, t) - d$. Their affine increments agree. The pullbacks of the closed Hilbert form of p_b have equal cell integrals: integrate its null first variation along the periodic difference of the two graphs. Subtracting their calibration identities leaves precisely the difference of the two squared defects. The kinetic integrals agree by translation on the rational cell. In the other terms change variables from x to $x + c$. This gives (B.18). All resulting integrands are periodic on that cell, because its horizontal deck increments and the corresponding vertical increments are integral. $\square$

Twice differentiating the case $b = a$ in y gives the leafwise matrix identity

$$(B.19) \quad - \int_{C_a} D_{x,u}^2 V(z_t(x)) dx = \int_{C_a} (D_{x,u} p_a)^T (D_{x,u} p_a)(z_t(x)) dx \succeq 0.$$

Its vertical entry equals the ground-state expression $\int_{C_a} |\nabla_x \log \phi_a|^2$, where $\phi_a = \partial_t U_a > 0$. Indeed differentiating $\nabla U_a = p_a(x, U_a)$ gives $\nabla \phi_a = (\partial_u p_a)(x, U_a) \phi_a$.

THEOREM B.10 (Exact effective-action defect). *Put $\bar{V} = \int_M V$, $\bar{p}_a = \int_M p_a$, and $B_a = \int_M |p_a|^2$. Every closed rational minimizing foliation satisfies*

$$(B.20) \quad \begin{aligned} \mathcal{A}(a) &= -\bar{V} - \frac{1}{2} B_a + a \cdot \bar{p}_a \\ &= \frac{1}{2} |a|^2 - \bar{V} - \frac{1}{2} \int_M |p_a - a|^2 dm. \end{aligned}$$

If $\mathcal{A}(a) = |a|^2/2 - \bar{V}$ for even one such slope, then $V_u = 0$.

Proof. Average (B.18) with $b = a$ over all ambient translations y . The left side becomes $\int_{C_a} V(z_t) dx - \bar{V}$. The right side becomes

$$\frac{1}{2} \int_{C_a} |p_a(z_t)|^2 dx + \frac{1}{2} B_a - a \cdot \bar{p}_a,$$

since $\int_{C_a} p_a(z_t) dx = \int_{C_a} \nabla U_a dx = a$. Subtract this identity from the kinetic part of the leaf action to obtain the first line of (B.20). Completing the square in the ambient average gives the second. Equality makes $p_a = a$ everywhere, hence the residual equation gives $Q = 0$. $\square$

The theorem does not assert that the upper envelope is touched at a finite slope. An escaping sequence with defect tending to zero is not such a touch. Theorem B.3 makes this distinction quantitative.

There is also an exact weighted Jacobi averaging identity. On each closed rational leaf,

$$\Delta_x \phi_a = Q_u(x, U_a) \phi_a, \quad \int_{C_a} \phi_a Q_u(x, U_a) dx = 0.$$

Thus the leaf transform $f \mapsto \int_{C_a} \phi_a f(x, U_a) dx$ annihilates Q_u . No injectivity theorem for the family of these transforms is assumed or inferred here.

Appendix C. Local identity-holonomy rigidity without a bandwidth restriction

This appendix uses closed rational leaves themselves, not a common volume. It is independent of the global common-volume theorem in Section 6. The additional hypothesis is smallness in an explicitly stated high-regularity norm. We do not remove it by a covering transformation.

THEOREM C.1 (Local rational-holonomy rigidity). *For every $n \geq 1$ there is $\delta_n > 0$ such that the following holds. Let $Q \in C^\infty(\mathbb{T}^{n+1})$ have zero vertical mean at every x . Suppose that, for each rational slope a , there is an ordered closed foliation by solutions of $\Delta U = Q(x, U)$ with identity adjusted rational holonomy. If*

$$\|Q\|_{C^{2n+6}} < \delta_n,$$

then $Q = 0$. In particular, for $Q = -V_u$ the conclusion is $V_u = 0$. No regularity in the slope parameter, Fourier-support restriction, or independent minimality assumption is needed.

C.1. A raw coefficient estimate. Fix $k \in \mathbb{Z}^n$, $\ell \in \mathbb{N}$, and $a = -k/\ell$. Work on $C_\ell = [0, \ell]^n$ with opposite faces identified. Identity adjusted holonomy gives

$$U(x + \ell e_i) = U(x) - k_i.$$

Write a leaf as $U(x) = a \cdot x + c + w_c(x)$ with w_c periodic and $\oint_{C_\ell} w_c = 0$. Every real c occurs. Indeed, if t is an ordered leaf parameter, the map

$$t \mapsto \oint_{C_\ell} (U(x, t) - a \cdot x) dx$$

is continuous and strictly increasing and increases by one when t increases by one. It is therefore onto. This uses no differentiation of the chosen leaf with respect to c .

Put $\epsilon = \|Q\|_\infty$ and $L = \|Q_u\|_\infty$. The leaf equation gives

$$(C.1) \quad \Delta w_c = Q(x, a \cdot x + c + w_c), \quad \|w_c\|_\infty \leq C_n \ell^2 \epsilon.$$

To justify the second assertion, rescale to the unit torus. Its zero-mean inverse Laplacian has an integrable Green kernel, so its map from zero-mean L^∞ to L^∞ is bounded. Rescaling back supplies exactly ℓ^2 . The right side has zero cell average because it equals Δw_c .

Averaging the equation and then using the Lipschitz constant L gives

$$(C.2) \quad \left| \oint_{C_\ell} Q(x, a \cdot x + c) dx \right| \leq C_n \ell^2 \epsilon L.$$

Let $H_a(c)$ denote the average inside this display. With the Fourier convention

$$\widehat{Q}(k, \ell) = \int_{\mathbb{T}^{n+1}} Q(x, u) e^{-2\pi i(k \cdot x + \ell u)} dx du,$$

a change of variable in the vertical integral yields

$$\begin{aligned} \int_0^1 H_a(c) e^{-2\pi i \ell c} dc &= \oint_{C_\ell} e^{2\pi i \ell a \cdot x} \left[\int_0^1 Q(x, u) e^{-2\pi i \ell u} du \right] dx \\ &= \widehat{Q}(k, \ell). \end{aligned}$$

The last integrand is unit periodic in x , so the normalized large-cell integral equals its unit-cell integral. Consequently

$$(C.3) \quad |\widehat{Q}(k, \ell)| \leq C_n \ell^2 \|Q\|_\infty \|Q_u\|_\infty.$$

Reality of Q gives the same bound for negative ℓ . No nonlinear phase character appears in this estimate.

C.2. Absorbing all Fourier modes. Let $d = n + 1$, $r = 2d + 4 = 2n + 6$, and $A = \|Q\|_{C^r}$. Suppose $Q \neq 0$, so $0 < \epsilon \leq A$. Zero vertical mean gives $\widehat{Q}(k, 0) = 0$. For an integer $N \geq 1$, split the absolutely convergent Fourier series. The low frequencies satisfy

$$\sum_{|(k, \ell)|_\infty \leq N} |\widehat{Q}(k, \ell)| \leq C_n N^{d+2} \epsilon L.$$

Integration by parts r times in a largest-frequency coordinate gives $|\widehat{Q}(k, \ell)| \leq C_{n,r} A (1 + |(k, \ell)|)^{-r}$, and therefore

$$\sum_{|(k, \ell)|_\infty > N} |\widehat{Q}(k, \ell)| \leq C_{n,r} A N^{d-r}.$$

It follows that

$$(C.4) \quad \epsilon \leq C_{n,r} N^{d+2} \epsilon L + C_{n,r} A N^{d-r}.$$

Choose

$$N = \left\lceil \left(\frac{2C_{n,r} A}{\epsilon} \right)^{1/(r-d)} \right\rceil,$$

with the fixed constant enlarged so that the second term is at most $\epsilon/2$. Since $A/\epsilon \geq 1$, the ceiling only changes a dimension-dependent multiplicative constant. Dividing by ϵ gives

$$(C.5) \quad \frac{1}{2} \leq C_{n,r} (A/\epsilon)^{(d+2)/(r-d)} L.$$

The periodic one-variable interpolation estimate is

$$L \leq C_r \epsilon^{1-1/r} A^{1/r}.$$

For clarity, apply at each fixed x the finite-difference inequality $\|Q_u\|_\infty \leq C_r (h^{-1} \epsilon + h^{r-1} \|\partial_u^r Q\|_\infty)$ and choose h a fixed multiple of $(\epsilon/A)^{1/r}$, within a fixed small interval. The estimate is uniform in x .

Set

$$\theta = 1 - \frac{1}{r} - \frac{d+2}{r-d} = \frac{3d+4}{(2d+4)(d+4)} > 0.$$

Substitution into (C.5) gives

$$\frac{1}{2} \leq C_{n,r} A^{1-\theta} \epsilon^\theta \leq C_{n,r} A.$$

Every nonzero force with the rational completeness hypothesis therefore has $A \geq c_n > 0$. Taking $\delta_n < c_n$ proves Theorem C.1. $\square$

COROLLARY C.2 (A finite rational obstruction at small amplitude). *For $Q \neq 0$ with $\|Q\|_{C^{2n+6}} < \delta_n$, some slope $-k/\ell$ with*

$$|k|_\infty, \ell \leq C_n \left(\frac{\|Q\|_{C^{2n+6}}}{\|Q\|_\infty} \right)^{1/(n+5)}$$

fails to possess an affine-periodic solution at every mean height on the cell of side ℓ .

Proof. If all those mean-height solutions existed, (C.3) would hold for every low mode used in (C.4); the same contradiction would follow. The proof requires only those solutions, not their ordering or smooth parameterization. $\square$

C.3. *The explicit boundary of the argument.* The exact vertical cover $V^{(N)}(x, u) = N^{-2}V(x, Nu)$ preserves closed rational foliations after rescaling slopes. Its force is $Q^{(N)} = N^{-1}Q(x, Nu)$. Although its uniform norm decreases,

$$\partial_u^r Q^{(N)} = N^{r-1}(\partial_u^r Q)(x, Nu).$$

Thus the high-regularity norm in Theorem C.1 generally grows. The theorem is not made global by that cover. No invariance of rational completeness under arbitrary amplitude interpolation $V \mapsto sV$ is assumed; Appendix K shows why individual rational foliations cannot be continued that way in general.

Appendix D. Constant anisotropic kinetic forms

Let G be a constant symmetric positive-definite matrix and replace (2.1) by

$$\mathcal{A}_{G,V}[v; \Omega] = \int_{\Omega} \left(\frac{1}{2} \nabla v^T G \nabla v - V(x, v) \right) dx.$$

The horizontal fields and their invariant-volume equations are unchanged. The residual equation becomes

$$(D.1) \quad \mathcal{R}_G(p) = \operatorname{div}_x(Gp) + \partial_u \left(\frac{1}{2} p^T G p \right) = Q.$$

Define

$$\mathcal{C}_{\mathbb{Q},G}(S) = \left\{ \beta \in \mathbb{Q}^n : (a - \beta)^T G (a - \beta) = (b - \beta)^T G (b - \beta) \text{ for all } a, b \in S \right\}.$$

THEOREM D.1 (Exact ellipsoidal criterion). *For a slope set containing at least two points, the common-volume universal rigidity theorem is true for $\mathcal{A}_{G,V}$ if and only if $\mathcal{C}_{\mathbb{Q},G}(S) = \emptyset$. If this centre set is nonempty, a nontrivial rational-shear potential has ordinary minimizing foliations at every slope and common-volume rigid foliations at every slope in S .*

Proof. The closed-phase construction uses only Frobenius and volume invariance, so remains valid. For two phases with common density, put $A = b - a - \nabla g$ as in Lemma 5.1. The residual difference is now

$$0 = \frac{\operatorname{div}_x(GA)}{q} + \frac{2A^T G p_{a,u}}{q} - \frac{A^T GA}{q^3} q_u.$$

Multiply by q and integrate over u . The last two terms integrate to zero. Thus $\operatorname{div}_x(G\nabla g) = 0$. Positive definiteness and periodic integration by parts give g constant. We again have $p_b - p_a = (b - a)/q$, with

$$(D.2) \quad 2q^2(b - a)^T G p_{a,u} = (b - a)^T G(b - a)q_u.$$

Write the common phase correction as ψ , and perform the same mean-removing shear as in Lemma 5.4. Substitution into (D.2), followed by the quotient integration in that lemma, gives

$$(D.3) \quad (Gh) \cdot \nabla_x \chi + \delta_G \chi_s = 0, \quad h = b - a, \quad \delta_G = \frac{b^T G b - a^T G a}{2}.$$

If $Q \neq 0$, then $\chi \neq 0$. Indeed $\chi = 0$ gives $q = 1$, $p_u = 0$, and Q independent of u ; its zero vertical mean then gives $Q = 0$. A nonzero Fourier coefficient (k, m) of χ has $m \neq 0$ and satisfies

$$h^T G k + \delta_G m = 0 \quad (b \in S).$$

Thus $\beta = -k/m$ is a rational centre in the G -metric. This proves sufficiency of the empty-centre condition.

For the converse, let $\beta = -k/\ell$ and let the common G -radius be $R > 0$. The autonomous construction applies with a vector e normalized by $e^T G e = 1$, because

$$\frac{1}{2} \xi^T G \xi + \frac{1}{2} b(u)^2 = \frac{1}{2} (\xi - e g_E(u))^T G (\xi - e g_E(u)) + g_E(u) (Ge) \cdot \xi - E.$$

The null term is the divergence of $(Ge)G_E(u)$. For a nonzero slope a , choose $T(E) = (a^T G a)^{-1/2}$ and $e = a/(a^T G a)^{1/2}$. The same profiles give all nonzero slopes; the zero-slope flow uses any such e and the coordinate $e \cdot x$. It remains bounded between consecutive zeros of b and is deck invariant.

Under $y = \ell u + k \cdot x$, the action transforms as in (3.5), with null term $-(Gk) \cdot \nabla y / \ell^2$ and constant $k^T G k / (2\ell^2)$. Hence the sine rational-shear potential again has ordinary foliations at all slopes. Choose $T(E) = 1/(\ell R)$. For every $(\alpha - \beta)^T G (\alpha - \beta) = R^2$, the phases (6.3) have the same $q = \ell R / g_E(\ell u + k \cdot x)$ and field $p_\alpha = \beta + (\alpha - \beta)/q$. The residual calculation with G gives $Q = -R^2 q_u / q^3 = -V_u$. The G -Hilbert form

$$\omega_{G,p} = \sum_i (Gp)_i du \wedge dx_i^\# - \left(\frac{1}{2} p^T G p + V \right) dx$$

is closed, with graph gap $\frac{1}{2}(\nabla v - p)^T G(\nabla v - p)$. This verifies class-A minimality directly. $\square$

If the selected slopes affinely span $\mathbb{R}^n$, the transport equations (D.3) give $\nabla_x \chi + \beta \chi_s = 0$, since G is invertible. The full-span classification is consequently unchanged except that the sphere is the G -ellipsoid and its radius is the G -radius. In the arbitrary-rank reconstruction the scalar Poisson equation is replaced by

$$\operatorname{div}_x(G\nabla f) = \operatorname{div}_x(GJ_a).$$

The residual under the shear is $\mathcal{R}_G(v_a) - \operatorname{div}_x(G\nabla f)$; the same calculation as in Theorem 7.1 proves the reconstruction. Constancy of G is essential to these identities. No analogous classification for a variable metric is asserted.

Appendix E. Uniformly convex kinetics and sharp universal slope tests

Let $n \geq 1$, $M = \mathbb{T}_x^n \times \mathbb{T}_u$, and normalize Haar measure $dm = dx du$ to have mass one. We study

$$(E.1) \quad \mathcal{A}_{K,V}[v; \Omega] = \int_{\Omega} (K(\nabla v) - V(x, v)) dx, \quad \operatorname{div} DK(\nabla v) = -V_u(x, v).$$

Here $V \in C^\infty(M)$ and $K \in C^\infty(\mathbb{R}^n)$ satisfies

$$(E.2) \quad D^2 K(p) \succeq \lambda_K I \quad \text{for all } p \in \mathbb{R}^n, \quad \lambda_K > 0.$$

The lower bound is global. It is stronger than pointwise strict convexity in some definitions of a Tonelli Lagrangian. This appendix concerns scalar graph variational problems, not an identification of their foliations with invariant Lagrangian tori of an ordinary-time Hamiltonian flow.

A smooth graph foliation of slope α consists of an ordered, deck-invariant family $U_\alpha(x, t)$ with $\partial_t U_\alpha > 0$, $U_\alpha(x, t+1) = U_\alpha(x, t) + 1$, and $U_\alpha(x, t) - \alpha \cdot x$ bounded in x . Its physical slope field is

$$p_\alpha(x, U_\alpha(x, t)) = \nabla_x U_\alpha(x, t).$$

It is smooth and periodic on M . A positive probability density q is a common invariant volume for a collection if

$$(E.3) \quad \partial_{x_i} q + \partial_u(q p_{\alpha,i}) = 0 \quad (1 \leq i \leq n, \alpha \in S), \quad \int_M q dm = 1.$$

All positive assertions below require only solution foliations. The counterexamples are minimizing against arbitrary multidimensional fixed-trace competitors.

THEOREM E.1 (Exact universal slope criterion). *Let $\emptyset \neq S \subset \mathbb{R}^n$. The following are equivalent.*

- (i) S is unbounded.

- (ii) For every K satisfying (E.2) and every smooth periodic V , the existence of smooth solution foliations of (E.1) at every slope in S , preserving one common smooth positive probability volume, implies $V_u = 0$.

For unbounded S one also has $q = 1$ and $p_\alpha = \alpha$ for every selected slope.

For bounded S , the counterexample can be chosen with two-sided global bounds $0 < \lambda I \preceq D^2 K \preceq \Lambda I$, quadratic growth, a real-analytic potential, and minimizing foliations at all selected slopes. It has ordinary smooth minimizing foliations at every real slope; every slope except one integer vector admits rigid holonomy.

The quantifier over K is essential. This theorem does not say that a fixed nonlinear K has no finite rigidity tests. It says that no bounded slope set is a sufficient test uniformly over the entire class (E.2). For a fixed Euclidean quadratic kinetic energy, the sharper arithmetic classification in Section 6 is not contradicted.

The symmetrized kinetic defect used below is already present in the kinetic version of the defect in Appendix B. We reconstruct it to make the present argument self-contained. The quantitative diameter estimate below has a converse over the entire kinetic class.

E.1. *From invariant volume to cohomological phases.*

LEMMA E.2. *For each foliation satisfying (E.3), there is a lifted phase*

$$(E.4) \quad \tau_\alpha(x, u) = u - \alpha \cdot x + \psi_\alpha(x, u), \quad \psi_\alpha \in C^\infty(M),$$

with

$$(E.5) \quad d\tau_\alpha = q(du - p_\alpha \cdot dx), \quad q = 1 + \partial_u \psi_\alpha, \quad qp_\alpha = \alpha - \nabla_x \psi_\alpha.$$

In particular, $\int_0^1 q(x, u) du = 1$ for every x , $\int_M qp_\alpha dm = \alpha$, and $\int_M q dm = 1$.

Proof. Integrability of the graph distribution gives

$$\partial_{x_i} p_j - \partial_{x_j} p_i + p_i \partial_u p_j - p_j \partial_u p_i = 0.$$

Together with (E.3), this makes the one-form $\theta = q(du - p \cdot dx)$ closed: the $dx_i \wedge du$ components are exactly (E.3), and substitution in the horizontal components gives

$$\partial_{x_i}(qp_j) - \partial_{x_j}(qp_i) = 0.$$

Also $m(x) = \int_0^1 q(x, u) du$ has zero gradient by (E.3); probability normalization makes $m = 1$. Thus θ has a lifted primitive of the form $u - \bar{\alpha} \cdot x + \psi$, with ψ periodic. It is constant on each lifted leaf. On a leaf of slope α , this identity and boundedness of ψ imply that $(\alpha - \bar{\alpha}) \cdot x$ is bounded on $\mathbb{R}^n$, so $\bar{\alpha} = \alpha$. This proves (E.4)–(E.5); their averages give the final statements. $\square$

The inverse family of (E.4) obeys

$$U_\alpha(x + m, t) = U_\alpha(x, t + \alpha \cdot m), \quad U_\alpha(x, t + 1) = U_\alpha(x, t) + 1.$$

Thus a smooth positive invariant density supplies the rigid normalization for each selected foliation. It does not make the densities furnished by different, independently normalized foliations equal.

Put $Q = -V_u$ and $H_K(p) = p \cdot DK(p) - K(p)$. The physical form of the equation is

$$(E.6) \quad \operatorname{div}_x DK(p_\alpha) + \partial_u H_K(p_\alpha) = Q.$$

Indeed, the derivative of H_K is $D^2K(p)p$, so its u derivative supplies exactly the derivative of the momentum along the graph in the vertical direction.

E.2. *The kinetic defect and a quantitative diameter bound.* Define

$$B_K(a, b) = K(a) - K(b) - DK(b) \cdot (a - b).$$

Uniform convexity gives $B_K(a, b) \geq \lambda_K |a - b|^2 / 2$.

LEMMA E.3 (Symmetrized kinetic defect). *Every phase in Lemma E.2 satisfies*

$$(E.7) \quad \begin{aligned} D &:= \int_M V(q - 1) dm = \int_M [B_K(\alpha, p_\alpha) + q B_K(p_\alpha, \alpha)] dm \\ &\geq \frac{\lambda_K}{2} \int_M (1 + q) |p_\alpha - \alpha|^2 dm. \end{aligned}$$

The left-hand side is the same number for every selected slope. If it vanishes for one phase, then $p_\alpha = \alpha$ and $V_u = 0$.

Proof. The convex perspective integrand is

$$f_{x,u}(\xi, \eta) = \eta K(-\xi/\eta) - V(x, u)\eta, \quad \eta > 0.$$

At $(\xi, \eta) = (-qp, q)$ its gradient is $(-DK(p), -H_K(p) - V)$. Equation (E.6) says that its divergence is zero. The affine phase $\tau^0 = u - \alpha \cdot x$ differs from τ_α by a periodic function, so integration of the linear term in the perspective Bregman identity gives

$$\mathcal{J}(\tau^0) - \mathcal{J}(\tau_\alpha) = \int_M B_K(\alpha, p_\alpha) dm.$$

Direct evaluation yields

$$K(\alpha) - \int_M q K(p_\alpha) dm + \int_M V(q - 1) dm = \int_M B_K(\alpha, p_\alpha) dm.$$

The cohomological averages give

$$\int_M q B_K(p_\alpha, \alpha) dm = \int_M q K(p_\alpha) dm - K(\alpha).$$

Combining these equalities proves (E.7). Equality and strict convexity give $p_\alpha = \alpha$, and then (E.6) gives $Q = 0$. $\square$

LEMMA E.4 (Cohomological separation). *For two selected slopes a, b , put $r_a = p_a - a$, $r_b = p_b - b$. There is a periodic $g : \mathbb{T}^n \rightarrow \mathbb{R}$ such that*

$$(E.8) \quad \|q(r_b - r_a)\|_{L^2(M)}^2 = |b - a|^2 \|q - 1\|_{L^2(M)}^2 + \|\nabla g\|_{L^2(\mathbb{T}^n)}^2.$$

Proof. Since $\partial_u \psi_a = \partial_u \psi_b$, their difference is $g(x)$. By (E.5),

$$q(p_b - p_a) = b - a - \nabla g, \quad q(r_b - r_a) = (b - a)(1 - q) - \nabla g.$$

Square and integrate in u . The cross term vanishes at each x because $\int_0^1 (1 - q) du = 0$. Integration in x gives (E.8). $\square$

THEOREM E.5 (Common-volume diameter bound). *Under (E.2)–(E.3), every pair $a, b \in S$ satisfies*

$$(E.9) \quad |b - a|^2 \|q - 1\|_{L^2(M)}^2 \leq \frac{8\|q\|_\infty^2}{\lambda_K} \int_M V(q - 1) dm.$$

If $V_u \neq 0$, then $q \neq 1$ and

$$\text{diam } S \leq \frac{2\sqrt{2}\|q\|_\infty}{\sqrt{\lambda_K}} \frac{\sqrt{\int_M V(q - 1) dm}}{\|q - 1\|_{L^2(M)}} < \infty.$$

Proof. By Lemma E.3, $\|r_a\|_2^2 \leq 2D/\lambda_K$ and the same holds for b . Hence

$$\|q(r_b - r_a)\|_2^2 \leq 2\|q\|_\infty^2 (\|r_a\|_2^2 + \|r_b\|_2^2) \leq 8\|q\|_\infty^2 D/\lambda_K.$$

Use Lemma E.4 and discard its nonnegative last term. If $q = 1$, then $D = 0$ and Lemma E.3 gives $V_u = 0$. This proves the stated nontrivial alternative. $\square$

For unbounded S , fix $a \in S$ and take $|b - a| \rightarrow \infty$ in (E.9). The finite right-hand side forces $q = 1$, and (E.7) gives $p_\alpha = \alpha$ for every selected slope and $V_u = 0$. This proves the sufficient direction of Theorem E.1. No compactness in slope, high-frequency asymptotic expansion, or slope derivative was used.

E.3. *Counterexamples for every bounded slope set.* The logarithm will be used only on a compact momentum interval. It will not be the kinetic energy at zero, at infinity, or at a singular endpoint.

LEMMA E.6 (Smooth uniformly convex logarithmic patch). *For every $0 < a < b < \infty$ there exists $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(s) = -\log s$ on $[a, b]$, $0 < \lambda \leq \varphi'' \leq \Lambda < \infty$, and φ is quadratic outside a compact set.*

Proof. Choose $0 \leq \chi \leq 1$ smooth, equal to one on $[a, b]$, with compact support in $(a/2, 2b)$. Set

$$\kappa(s) = 1 - \chi(s) + \frac{\chi(s)}{s^2},$$

where the last term is extended by zero outside the positive support. It is smooth and bounded above and below by positive constants. Integrate $\varphi'' = \kappa$, prescribing $\varphi(s_0) = -\log s_0$ and $\varphi'(s_0) = -1/s_0$ for one $s_0 \in [a, b]$. On that interval the value, first derivative, and second derivative agree with $-\log s$. Outside the support, $\varphi'' = 1$. $\square$

Fix bounded nonempty S . Choose $\beta \in \mathbb{Z}^n$ with

$$\beta_i < \inf_{\alpha \in S} \alpha_i \quad (1 \leq i \leq n).$$

Then all coordinates of $\gamma = \alpha - \beta$ lie in one compact subinterval of $(0, \infty)$. Fix $\varepsilon > 0$ and define

$$(E.10) \quad b(s) = \sin^2(\pi s), \quad Z_\varepsilon = \int_0^1 e^{-\varepsilon b(s)^2} ds, \quad q_0(s) = Z_\varepsilon^{-1} e^{-\varepsilon b(s)^2}.$$

Let $F(s) = \int_0^s q_0(t) dt$. It is an increasing smooth degree-one diffeomorphism. Select $[a, b] \Subset (0, \infty)$ containing every number $\gamma_i/q_0(s)$ with $\alpha \in S$ and $s \in \mathbb{R}$, and choose φ from Lemma E.6. Put

$$(E.11) \quad K_0(\xi) = \sum_{i=1}^n \varphi(\xi_i), \quad K(p) = K_0(p - \beta), \quad V(x, u) = n \log q_0(u - \beta \cdot x).$$

This K has two-sided positive Hessian bounds and quadratic growth. The potential is smooth, real analytic, periodic, and nonconstant in u .

PROPOSITION E.7 (The selected common-volume families). *For every $\alpha \in S$, the leaves*

$$(E.12) \quad U_\alpha(x, t) = \beta \cdot x + F^{-1}((\alpha - \beta) \cdot x + t)$$

are class-A minimizers for (E.11), have slope α , and have rigid deck law. They all preserve the positive probability volume

$$q(x, u) dm = q_0(u - \beta \cdot x) dx du.$$

Proof. The inverse phases and fields are

$$\tau_\alpha = F(u - \beta \cdot x) - (\alpha - \beta) \cdot x, \quad p_\alpha = \beta + \frac{\alpha - \beta}{q_0(u - \beta \cdot x)}.$$

Their deck increments are $r - \alpha \cdot m$, and $(\tau_\alpha)_u = q$. In particular, they have the asserted common invariant density. The difference $U_\alpha - \alpha \cdot x$ is bounded because $F^{-1}(s) - s$ is periodic.

For minimality, make the integer shear $v = u - \beta \cdot x$. The action becomes

$$\int_\Omega (K_0(\nabla v) - V_0(v)) dx, \quad V_0(v) = n \log q_0(v).$$

For $\gamma = \alpha - \beta$ write $P_\gamma(v) = \gamma/q_0(v)$. All its coordinates lie in the logarithmic patch, so

$$K_0(P_\gamma) = - \sum_i \log \gamma_i + n \log q_0(v), \quad (DK_0(P_\gamma))_i = - \frac{q_0(v)}{\gamma_i}.$$

Consequently

$$(E.13) \quad P_\gamma \cdot DK_0(P_\gamma) - K_0(P_\gamma) + V_0 = \sum_i \log \gamma_i - n =: E_\gamma.$$

For every competitor w there is the pointwise identity

$$(E.14) \quad K_0(\nabla w) - V_0(w) = B_{K_0}(\nabla w, P_\gamma(w)) + DK_0(P_\gamma(w)) \cdot \nabla w - E_\gamma.$$

The middle term is $-\sum_i \gamma_i^{-1} \partial_{x_i} F(w)$, a divergence. Its integral agrees for fixed-trace competitors. On (E.12), $\nabla v = P_\gamma(v)$, so its Bregman term vanishes. Thus the action difference is precisely the integral of the nonnegative Bregman term in (E.14). This controls competitors depending on every base variable. Quadratic growth and bounded derivatives of F justify the same identity for $W^{1,2}$ fixed-trace competitors by the Sobolev chain rule.

The equation can also be checked directly: on a leaf,

$$\sum_i \partial_{x_i} \left(- \frac{q_0(v)}{\gamma_i} \right) = -n \frac{q'_0(v)}{q_0(v)} = -V'_0(v).$$

□

This already proves the necessary direction of Theorem E.1. In particular, it disproves the unchanged Euclidean rational-centre criterion for the whole nonlinear kinetic class: take $S = \{0, e_1, 2e_1\}$. No sphere contains these three distinct collinear points, yet Proposition E.7 supplies common-volume minimizers with $V_u \neq 0$.

E.3.1. Retaining ordinary foliations at all slopes. We verify the additional all-slope assertion rather than restricting the converse to a finite or bounded set.

Fix a unit vector e and define

$$h_e(r) = r DK_0(re) \cdot e - K_0(re), \quad r \geq 0.$$

Then

$$h_e(0) = -K_0(0), \quad h'_e(r) = r e^T D^2 K_0(re) e,$$

and therefore

$$\frac{\lambda}{2} r^2 \leq h_e(r) + K_0(0) \leq \frac{\Lambda}{2} r^2.$$

Let $E_* = \max V_0 - K_0(0)$. For $E > E_*$ there is a unique smooth positive periodic $g_{E,e}$ satisfying

$$(E.15) \quad h_e(g_{E,e}(v)) + V_0(v) = E.$$

Its time period $T_e(E) = \int_0^1 g_{E,e}(v)^{-1} dv$ is continuous and strictly decreasing. It tends to zero as $E \rightarrow \infty$.

At $E = E_*$, the nonnegative profile $g_{*,e}$ satisfies

$$h_e(g_{*,e}) + K_0(0) = \max V_0 - V_0 = n\epsilon b(v)^2.$$

Since $h_e(r) + K_0(0) = r^2 R_e(r)$ with R_e smooth and positive for $r \geq 0$, the inverse-function theorem applied to $r\sqrt{R_e(r)}$ at zero shows that $g_{*,e}$ is smooth through the zeros of b . It is bounded above and below by positive multiples of b near those zeros. Thus $\int_0^1 g_{*,e}^{-1} = \infty$, and monotone convergence gives $T_e(E) \rightarrow \infty$ as $E \downarrow E_*$. For every $\gamma \neq 0$, choose $e = \gamma/|\gamma|$ and the unique E for which $T_e(E) = 1/|\gamma|$. The normalized primitive of $1/g_{E,e}$ and its inverse give a rigid foliation with slope γ solving $\nabla v = eg_{E,e}(v)$.

For $\gamma = 0$, fix one e and let Φ_s be the complete flow of $\dot{v} = g_{*,e}(v)$. The family $v(x, t) = \Phi_{e \cdot x}(t)$ is smooth, ordered, vertically periodic, and ordinary deck invariant. Its leaves remain between consecutive integer equilibria, so their slope is zero.

Every one of these families is minimizing: for $P(v) = eg_{E,e}(v)$, equation (E.15) gives

$$K_0(\xi) - V_0(v) = B_{K_0}(\xi, P(v)) + DK_0(P(v)) \cdot \xi - E.$$

The momentum depends only on v , hence has a vector-valued primitive and gives a fixed-boundary divergence. This is the same all-dimensional comparison used in Proposition E.7.

Finally there is no alternative closed zero-slope minimizing foliation. On any periodic cover, the action excess relative to a constant maximizing V_0 is

$$\int [B_{K_0}(\nabla v, 0) + \max V_0 - V_0(v)] dx,$$

since the integral of $DK_0(0) \cdot \nabla v$ is zero. A periodic class-A minimizer is a cell minimizer: tiling a lower-energy cell competitor over a large cube and joining it in a fixed-width collar contradicts class-A minimality because the volume gain dominates the boundary cost. Thus both displayed nonnegative terms must vanish. Such a minimizer is constant and takes an integer value. These constants cannot fill the circle. Undoing the integer shear proves that β is the unique exceptional slope for existence of rigid minimizing foliations.

E.4. What replaces the quadratic phase-locking calculation. For arbitrary autonomous K , the continuity equation and (E.6) imply the exact weighted conservation law

$$(E.16) \quad qQ = \operatorname{div}_x(qDK(p_\alpha)) + \partial_u(qp_\alpha \cdot DK(p_\alpha)).$$

It follows by expanding the right-hand side and using $\nabla_x q = -p_\alpha q_u - q\partial_u p_\alpha$. For two common-volume phases, $p_b - p_a = (h - \nabla g)/q$ with $h = b - a$. Subtract

(E.16) and integrate in u to obtain

$$(E.17) \quad \operatorname{div}_x (M_{ab}(x)(h - \nabla g)) = 0,$$

where

$$M_{ab}(x) = \int_0^1 \int_0^1 D^2 K \left(p_a(x, u) + t \frac{h - \nabla g(x)}{q(x, u)} \right) dt du.$$

For quadratic K , M_{ab} is a constant positive matrix and periodicity forces g to be constant. For nonlinear K , the matrix generally depends on x and on the corrector. Equation (E.17) does not justify replacing it by a constant matrix. The universal theorem above avoids that replacement entirely. No claim that (E.17) classifies all finite compatible families for a fixed nonlinear K is made.

E.5. *An inverse-square estimate for an individual kinetic phase.* Write $\kappa = \lambda_K > 0$ for the global convexity constant. The symmetrized defect gives

$$(E.18) \quad \int_M V(q_\alpha - 1) dm \geq \frac{\kappa}{2} \int_M (1 + q_\alpha) |p_\alpha - \alpha|^2 dm.$$

PROPOSITION E.8 (Quantitative extension with bounded Hessian). *Assume in addition $D^2 K \preceq LI$ globally for some finite L . If $Q \neq 0$, there is $c_{K,V} > 0$ such that*

$$(E.19) \quad \int_M V(q_\alpha - 1) dm \geq \frac{c_{K,V}}{1 + |\alpha|^2}$$

for every admissible phase. The ensemble escape conclusions follow unchanged.

Proof. Put $r = p - \alpha$, $A_p = DK(p) - DK(\alpha)$ and $D = \int V(q - 1)$. The exact identity

$$H(p) - H(\alpha) = \alpha \cdot A_p + B_K(\alpha, p)$$

rewrites (E.6) as

$$Q = \operatorname{div}_x A_p + \partial_u (\alpha \cdot A_p + B_K(\alpha, p)).$$

Choose a nonzero Fourier coefficient $c = |\widehat{Q}(k, \ell)|$, $\ell \neq 0$. Since $|A_p| \leq L|r|$, (E.18) and (E.7) give

$$c \leq 2\pi L \sqrt{2/\kappa} |k + \ell\alpha| \sqrt{D} + 2\pi |\ell| D.$$

The proof of Theorem B.2 now applies with $A = 2\pi L \sqrt{2/\kappa} \sqrt{|k|^2 + \ell^2}$ and $B = 2\pi |\ell|$. $\square$

The global upper Hessian bound is used only in this quantitative extension. The exact separation identity needs no such bound. Neither theorem asserts the rational-sphere classification for a general nonlinear K ; that classification uses the quadratic residual difference.

Appendix F. Nonlinear elliptic selection on a fixed torus

Let $n \geq 1$, $M = \mathbb{T}_x^n \times \mathbb{T}_u$, and let m be normalized Haar measure. Consider

$$(F.1) \quad \mathcal{A}_{K,V}[v; \Omega] = \int_{\Omega} (K(\nabla v) - V(x, v)) \, dx, \quad Q = -V_u,$$

where $V \in C^\infty(M)$ and

$$(F.2) \quad K \in C^\infty(\mathbb{R}^n), \quad \lambda I \preceq D^2 K(p) \preceq \Lambda I \quad (p \in \mathbb{R}^n), \quad 0 < \lambda \leq \Lambda < \infty.$$

Analytic conclusions will additionally require $K \in C^\omega(\mathbb{R}^n)$. No analyticity in x or u is imposed on V or on the prescribed density.

A class-A minimizer minimizes (F.1) against all $W_0^{1,2}(\Omega)$ perturbations on every bounded Lipschitz domain. A smooth graph foliation has a smooth, periodic slope field $p(x, u)$: the graph through (x, u) has gradient $p(x, u)$. A positive probability volume $q \, dm$ is invariant under its horizontal fields if

$$(F.3) \quad \partial_{x_i} q + \partial_u (q p_i) = 0, \quad 1 \leq i \leq n.$$

The common-volume condition means that the same q works for all selected foliations, not that their independently chosen densities are equal by convention.

Fix a smooth $q > 0$ satisfying

$$(F.4) \quad \int_0^1 q(x, u) \, du = 1 \quad (x \in \mathbb{T}^n).$$

This normalization follows from (F.3) and probability normalization whenever even one invariant graph foliation exists. Define

$$(F.5) \quad \psi_u = q - 1, \quad \int_0^1 \psi(x, u) \, du = 0, \quad f(x) = \int_0^1 q(x, u) Q(x, u) \, du.$$

A necessary condition for compatibility is $\int_{\mathbb{T}^n} f = 0$; if it fails, the compatible slope set is empty. Henceforth assume this condition.

Definition F.1. The compatibility spectrum $\Sigma_{K,V}(q)$ is the set of slopes $a \in \mathbb{R}^n$ admitting a smooth solution graph foliation for (F.1) whose horizontal fields preserve $q \, dm$.

Such foliations are automatically class-A minimizing, by the convex Hilbert calibration proved below. The existence of the invariant volume also supplies the rigid inverse-phase cohomology; rigid holonomy is not a second hidden assumption in this definition.

For $a \in \mathbb{R}^n$ and a mean-zero $g \in H^1(\mathbb{T}^n)$ put

$$(F.6) \quad P_{a,g}(x, u) = \frac{a - \nabla_x \psi(x, u) - \nabla g(x)}{q(x, u)}$$

and define the auxiliary functional

$$(F.7) \quad \mathcal{E}_a(g) = \int_M q^2 K(P_{a,g}) dm - \int_{\mathbb{T}^n} f(x)g(x) dx.$$

The factor q^2 is essential.

THEOREM F.2 (Exact elliptic selector). *For every $a \in \mathbb{R}^n$, (F.7) has a unique mean-zero minimizer g_a , and g_a is smooth. It is the unique smooth mean-zero solution of*

$$(F.8) \quad \operatorname{div}_x \left[\int_0^1 q DK(P_{a,g_a}) du \right] = f.$$

Put $P_a = P_{a,g_a}$ and

$$(F.9) \quad \mathcal{R}(a) = \operatorname{div}_x DK(P_a) + \partial_u(P_a \cdot DK(P_a) - K(P_a)) - Q.$$

Then

$$(F.10) \quad a \in \Sigma_{K,V}(q) \iff \mathcal{R}(a) \equiv 0.$$

For a compatible slope, its slope field is uniquely P_a , and an inverse phase is

$$(F.11) \quad \tau_a = u - a \cdot x + \psi + g_a.$$

The maps $a \mapsto g_a$ and $a \mapsto \mathcal{R}(a)$ are smooth. If K is real analytic, they are real analytic in fixed Hölder spaces.

The selector exists at *every* slope, including incompatible slopes. Outside (F.10) it gives a graph foliation but not a foliation of solutions. This distinction is what makes the parameter argument legitimate: no vanishing obstruction is inferred from solvability of the auxiliary equation.

THEOREM F.3 (Analytic compatibility spectrum). *Assume additionally that K is real analytic. There is a nonnegative real-analytic function $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$(F.12) \quad \Sigma_{K,V}(q) = \{a : \mathcal{F}(a) = 0\}.$$

Precisely one of the following alternatives holds:

- (i) $q \equiv 1$, $Q \equiv 0$, and $\Sigma_{K,V}(q) = \mathbb{R}^n$;
- (ii) $\Sigma_{K,V}(q)$ is a compact proper real-analytic zero set, possibly empty.

In the second alternative, $\Sigma_{K,V}(q)$ has zero n -dimensional Lebesgue measure. Every affine line meets $\Sigma_{K,V}(q)$ in finitely many points. In fact there is a finite bound on that number depending on K, V, q , uniform over affine lines.

COROLLARY F.4 (Two exact sufficient configurations). *For a real-analytic kinetic energy satisfying (F.2), common-volume solution foliations force $q = 1$ and $V_u = 0$ if their slope set either has positive n -dimensional Lebesgue measure*

or contains infinitely many distinct collinear points. The collinear set may be bounded and may consist entirely of rational slopes.

No universal finite cardinality is asserted here. The upper bound in Theorem F.3 depends on the full fixed data.

F.1. *Relation to the Euclidean calculation.* The selector replaces the harmonic horizontal correction in the quadratic calculation by a nonlinear elliptic one. Its residual is retained: solving the selector equation is not the same as solving the foliation equation. The variational, calibration, and implicit-function ingredients are classical; compare [5] and [7].

F.2. *Closed phases and the weighted residual.* Write

$$(F.13) \quad H_K(p) = p \cdot DK(p) - K(p), \quad R_K(p) = \operatorname{div}_x DK(p) + \partial_u H_K(p).$$

The equation along every graph of a foliation is equivalent to $R_K(p) = Q$. Indeed, total differentiation of $DK(\nabla U)$ adds $p \cdot D^2 K(p)p_u$ to its physical x -divergence, which is precisely $\partial_u H_K(p)$.

LEMMA F.5 (Closed phase and slope cohomology). *Let a smooth graph foliation of slope a preserve $q \, dm$. Then $q(du - p \cdot dx)$ is closed and has a primitive*

$$(F.14) \quad \tau = u - a \cdot x + \psi + g(x),$$

where ψ is as in (F.5) and g is periodic, unique up to a constant. In particular

$$(F.15) \quad \int_M q \, dm = 1, \quad \int_M qp \, dm = a.$$

Proof. Graph integrability gives $\partial_i p_j - \partial_j p_i + p_i \partial_u p_j - p_j \partial_u p_i = 0$. Combining this with (F.3) proves the horizontal components of closedness; the mixed components are (F.3) itself. The vertical period is one. Therefore a primitive has the form $u - c \cdot x +$ a periodic function. On each leaf it is constant. Comparing this identity with bounded deviation of the leaf from $a \cdot x$ gives $c = a$, since $(a - c) \cdot x$ cannot be bounded on $\mathbb{R}^n$ unless $a = c$. Its vertical derivative fixes the periodic part up to $g(x)$. Integrating $qp = a - \nabla \psi - \nabla g$ gives (F.15). $\square$

LEMMA F.6 (Weighted residual identity). *Whenever (F.3) holds,*

$$(F.16) \quad qR_K(p) = \operatorname{div}_x(qDK(p)) + \partial_u(qp \cdot DK(p)).$$

Consequently every compatible phase satisfies (F.8) and $\int f = 0$.

Proof. The continuity equations give $\nabla_x q = -q_u p - qp_u$. Expand the right side of (F.16). The terms $(\nabla_x q) \cdot DK(p) + q_u p \cdot DK(p) + qp_u \cdot DK(p)$ cancel. The remaining terms are $q \operatorname{div}_x DK(p) + qp \cdot D^2 K(p)p_u = qR_K(p)$. Vertical integration gives the selector equation; horizontal integration gives $\int f = 0$. $\square$

F.3. *Construction, coercivity, and exact reconstruction.* For fixed a , let

$$(F.17) \quad J_a(x, \xi) = \int_0^1 q(x, u)^2 K\left(\frac{a - \nabla\psi(x, u) - \xi}{q(x, u)}\right) du.$$

Then $\mathcal{E}_a(g) = \int J_a(x, \nabla g) - fg$ and

$$(F.18) \quad D_\xi J_a = - \int_0^1 q DK(P_{a,g}) du, \quad D_{\xi\xi}^2 J_a = \int_0^1 D^2 K(P_{a,g}) du.$$

In particular,

$$(F.19) \quad \lambda I \preceq D_{\xi\xi}^2 J_a \preceq \Lambda I.$$

The constants are independent of the slope and no rational cover occurs.

Proof of the existence and uniqueness in Theorem F.2. For a mean-zero function g , Taylor's inequality at $\xi = 0$, followed by Poincaré and Young inequalities, gives

$$\mathcal{E}_a(g) \geq (\lambda/4) \|\nabla g\|_2^2 - C_a.$$

The direct method gives a minimizer in H^1 ; strict convexity in the gradient and the zero-mean gauge give uniqueness. Its weak first variation is

$$(F.20) \quad - \int_M q DK(P_{a,g_a}) \cdot \nabla \eta \, dm - \int_{\mathbb{T}^n} f \eta = 0.$$

A possible constant multiplier for the mean constraint is zero because $\int f = 0$. Thus (F.8) holds distributionally.

The regularity lemma in Appendix O applies. To check its only nontrivial growth condition, differentiate (F.18) in x at fixed ξ . On compact slope sets, boundedness of q, q^{-1}, ψ and their derivatives, together with $|DK(p)| \leq |DK(0)| + \Lambda|p|$, gives

$$(F.21) \quad |D_x D_\xi J_a(x, \xi)| \leq C(1 + |\xi|).$$

Hence g_a is smooth, with bounds in every fixed Hölder space locally uniform in a . $\square$

PROPOSITION F.7 (Exact variational projection). *For every mean-zero $g \in H^1(\mathbb{T}^n)$,*

$$(F.22) \quad \mathcal{E}_a(g) - \mathcal{E}_a(g_a) = \int_M q^2 B_K(P_{a,g}, P_a) \, dm \geq \frac{\lambda}{2} \|\nabla(g - g_a)\|_2^2,$$

where $B_K(v, w) = K(v) - K(w) - DK(w) \cdot (v - w)$.

Proof. Subtract the two energies and their first-order Taylor terms at g_a . The latter vanish by (F.20). Since $P_{a,g} - P_a = -\nabla(g - g_a)/q$, the lower Hessian bound cancels the factor q^2 and gives (F.22). $\square$

Proof of the exact characterization (F.10). A compatible foliation has the form of Lemma F.5. Lemma F.6 makes its mean-zero g a solution of the selector equation, hence $g = g_a$. This proves necessity and uniqueness.

Conversely, define (F.11). Its vertical derivative is $q > 0$, so it is invertible on each vertical line and its level graphs form a smooth foliation. The periodic correction gives

$$(F.23) \quad U_a(x, t+1) = U_a(x, t) + 1, \quad U_a(x+j, t) = U_a(x, t+a \cdot j), \quad j \in \mathbb{Z}^n.$$

Also $U_a - a \cdot x$ is bounded on each leaf, and its slope field is P_a . If $\mathcal{R}(a) = 0$, its leaves solve the original Euler equation.

Let $dx_i^\# = \iota_{\partial_{x_i}}(dx_1 \wedge \cdots \wedge dx_n)$. The form

$$(F.24) \quad \omega_a = \sum_i DK(P_a)_i du \wedge dx_i^\# - (H_K(P_a) + V) dx$$

is closed exactly when $R_K(P_a) + V_u = 0$. For every graph v ,

$$(F.25) \quad (K(\nabla v) - V(x, v)) dx = v^* \omega_a + B_K(\nabla v, P_a(x, v)) dx.$$

The last term is nonnegative and vanishes on each selected leaf. Integrals of the closed form agree for fixed-boundary graphs on the universal cover. Thus the leaves are class-A minimizing against arbitrary n -dimensional competitors. Strong $W^{1,2}$ approximation extends the identity from smooth competitors: the two-sided Hessian bounds give quadratic growth, and the periodic coefficients in the form are bounded with bounded derivatives. $\square$

F.4. *Parameter dependence without assuming solution foliations.* Fix $0 < \theta < 1$. Use the fixed spaces

$$(F.26) \quad X = \{g \in C^{2,\theta}(\mathbb{T}^n) : \int g = 0\}, \quad Y = \{h \in C^{0,\theta}(\mathbb{T}^n) : \int h = 0\}.$$

Let $\mathcal{G}(a, g)$ be the left side of (F.8) minus f . At $g = g_a$,

$$(F.27) \quad D_g \mathcal{G}(a, g_a) \eta = -\operatorname{div}_x(M_a \nabla \eta), \quad M_a(x) = \int_0^1 D^2 K(P_a(x, u)) du.$$

This is an isomorphism $X \rightarrow Y$. Its kernel before mean normalization consists exactly of constants. Its L^2 coercivity is

$$(F.28) \quad \int_{\mathbb{T}^n} M_a \nabla \eta \cdot \nabla \eta dx \geq 4\pi^2 \lambda \|\eta\|_2^2 \quad \left(\int \eta = 0 \right).$$

Lax–Milgram and Schauder theory give surjectivity in (F.26).

If K is real analytic, $\mathcal{G}$ is real analytic near every (a, g_a) . Indeed the momentum range on the compact torus is compact; real analyticity of K gives uniform local factorial bounds there. Pointwise composition is consequently analytic between the Hölder spaces used in (F.26). The analytic implicit-function theorem gives a local analytic branch. Uniqueness patches those branches over

all of $\mathbb{R}^n$. The same argument with smooth composition gives the smooth assertion when K is merely smooth.

This proves the parameter part of Theorem F.2. In particular $a \mapsto \mathcal{R}(a)$ is analytic into $C^{0,\theta}(M)$ in the analytic case. Define

$$(F.29) \quad \mathcal{F}(a) = \int_M |\mathcal{R}(a, x, u)|^2 dm.$$

It is nonnegative and real analytic, and its zero set is exactly (F.10).

Remark F.8 (What has, and has not, been continued). The analytic family is the minimizer of (F.7), not a presumed family of solutions of the original foliation equation. The original equation has the generally nonzero residual (F.9). The selector construction therefore does not make the invalid inference that solvability of a range equation kills an obstruction.

Appendix G. Generated forces, properness, and analytic compatibility

G.1. *Coercivity of the compatibility spectrum.* We recall the kinetic defect because it is the coercive ingredient needed for compactness. Define

$$(G.1) \quad D = \int_M V(q - 1) dm.$$

This number is independent of the selected slope.

LEMMA G.1 (Symmetrized kinetic defect). *At every compatible slope a ,*

$$(G.2) \quad D = \int_M [B_K(a, P_a) + qB_K(P_a, a)] dm \geq \frac{\lambda}{2} \int_M (1 + q)|P_a - a|^2 dm.$$

Proof. Apply Lemma E.3 to the compatible phase τ_a . The common density makes its left-hand side independent of a . $\square$

For two compatible slopes a, b , put $r_a = P_a - a$ and $h = g_b - g_a$. Subtracting their exact phases gives

$$(G.3) \quad q(r_b - r_a) = (b - a)(1 - q) - \nabla h.$$

The cross term disappears on squaring and integrating because of (F.4). Therefore

$$(G.4) \quad \|q(r_b - r_a)\|_2^2 = |b - a|^2 \|q - 1\|_2^2 + \|\nabla h\|_2^2.$$

Together with (G.2), this gives

$$(G.5) \quad |b - a|^2 \|q - 1\|_2^2 \leq \frac{8\|q\|_\infty^2 D}{\lambda}.$$

The separation estimate appeared in Appendix E; its present role is to compactify the exact residual zero set.

PROPOSITION G.2. *If $\Sigma_{K,V}(q)$ is nonempty and $q \neq 1$, then $D \geq 0$ and $\Sigma_{K,V}(q)$ is bounded. If $q \equiv 1$, then $\Sigma_{K,V}(q)$ is empty when $Q \neq 0$, and equals $\mathbb{R}^n$ when $Q = 0$.*

Proof. For nonconstant q , fix $a \in \Sigma_{K,V}(q)$ and use (G.5) for every $b \in \Sigma_{K,V}(q)$. If $q = 1$, then $\psi = 0$ and $f = 0$. The unique selector is $g_a = 0$: the constant flux $DK(a)$ solves its equation. Thus $P_a = a$ and $\mathcal{R}(a) = -Q$. $\square$

Closedness follows from continuity of the residual. This proves the compactness alternative in Theorem F.3, even for smooth K .

G.2. *A universal source normal form, independent of the target potential.* There is a complementary selector with a useful additional property: it does not depend on the target V at all. It is the partial minimization of the *actual* perspective action, rather than of its weighted residual projection. The two selectors agree at compatible slopes; they are not identified away from the compatibility set.

Fix K, q, ψ as above, but impose no condition involving Q or f . For every $a \in \mathbb{R}^n$, let h_a be the unique mean-zero minimizer of

$$(G.6) \quad \mathcal{H}_a(h) = \int_M qK \left(\frac{a - \nabla\psi - \nabla h}{q} \right) dm, \quad \widehat{P}_a = \frac{a - \nabla\psi - \nabla h_a}{q}.$$

Its equation is

$$(G.7) \quad \operatorname{div}_x \int_0^1 DK(\widehat{P}_a) du = 0.$$

Define the force and its zero-vertical-mean potential by

$$(G.8) \quad Q_q(a) = R_K(\widehat{P}_a), \quad -\partial_u V_q(a) = Q_q(a), \quad \int_0^1 V_q(a, x, u) du = 0.$$

THEOREM G.3 (Universal source normal form and properness). *The minimizer h_a exists uniquely and is smooth for every a . The force in (G.8) has zero vertical mean at every x , so $V_q(a)$ is well defined and smooth. The phases*

$$\widehat{\tau}_a = u - a \cdot x + \psi + h_a$$

are rigid minimizing foliations for the potentials $V_q(a)$.

Every smooth solution foliation with slope a , preserving $q dm$ for the kinetic energy K , is this foliation up to an additive phase constant, and its potential equals $V_q(a) + A(x)$. Hence for any prescribed V ,

$$(G.9) \quad \Sigma_{K,V}(q) = \{a : Q_q(a) = -V_u\}.$$

If K is real analytic, $a \mapsto Q_q(a)$ and $a \mapsto V_q(a)$ are real analytic in fixed Hölder spaces.

When $q \neq 1$, define

$$(G.10) \quad \begin{aligned} \delta^2 &= \int_M (q-1)^2 dm, & b_q &= \delta^{-2} \int_M (1-q) \nabla \psi dm, \\ E_q &= \int_M |\nabla \psi - b_q(1-q)|^2 dm. \end{aligned}$$

Then every a satisfies the exact orthogonal decomposition

$$(G.11) \quad \|q(\widehat{P}_a - a)\|_2^2 = \delta^2 |a - b_q|^2 + E_q + \|\nabla h_a\|_2^2$$

and the coercive estimate

$$(G.12) \quad \|Q_q(a)\|_\infty \geq \|V_q(a)\|_\infty \geq \frac{\lambda(\delta^2 |a - b_q|^2 + E_q + \|\nabla h_a\|_2^2)}{2\|q\|_\infty^2 \|q-1\|_1}.$$

In particular the source map is proper into $C^0(M)$: inverse images of compact sets are compact. Every fixed-source fibre is bounded by an explicit ball centred at b_q .

Proof. The Hessian in ∇h of the integrand in (G.6) is

$$\int_0^1 q^{-1} D^2 K(\widehat{P}_a) du.$$

It lies between $(\lambda/\|q\|_\infty)I$ and $(\Lambda/\min q)I$. Its mixed x -gradient derivative has at most linear growth in ∇h . The same direct method, regularity lemma, and fixed-space implicit-function argument used for g_a give existence, uniqueness, smoothness, and analytic dependence. An exact projection identity also holds:

$$\mathcal{H}_a(h) - \mathcal{H}_a(h_a) = \int_M q B_K(P_{a,h}, \widehat{P}_a) dm \geq \frac{\lambda}{2\|q\|_\infty} \|\nabla(h - h_a)\|_2^2.$$

The mean of $R_K(\widehat{P}_a)$ in u is $\operatorname{div}_x \int DK(\widehat{P}_a) du = 0$, by (G.7). Thus (G.8) defines a periodic potential. The exact phase and Hilbert calibration then prove that its leaves are minimizing.

Conversely, for any solution foliation with the prescribed volume, integrating its full equation in u gives (G.7), because $\int Q du = 0$. Uniqueness forces its horizontal correction to equal h_a . The force, and therefore the potential modulo $A(x)$, are determined. This proves completeness of the normal form.

For the orthogonal identity write $r_a = \widehat{P}_a - a$. Then

$$qr_a = a(1-q) - \nabla \psi - \nabla h_a.$$

The first two terms have zero mean in u for every x , whereas ∇h_a is independent of u . Their cross term vanishes. The definition of b_q makes $\nabla \psi - b_q(1-q)$ orthogonal in $L^2(M; \mathbb{R}^n)$ to every constant-vector multiple of $1-q$. Completing the square proves (G.11).

Apply the symmetrized kinetic defect to the actual potential $V_q(a)$ just constructed:

$$\int_M V_q(a)(q-1) dm \geq \frac{\lambda}{2} \|r_a\|_2^2 \geq \frac{\lambda}{2\|q\|_\infty^2} \|qr_a\|_2^2.$$

The left side is bounded above by $\|V_q(a)\|_\infty \|q-1\|_1$, proving the second inequality in (G.12). A zero-mean periodic primitive satisfies $\|V_q(a, x, \cdot)\|_\infty \leq \|\partial_u V_q(a, x, \cdot)\|_\infty$; this proves the first inequality. If a set of forces is bounded in C^0 , (G.12) bounds all its preimages in $\mathbb{R}^n$. Continuity gives closedness of the preimage of a compact set; finite-dimensional compactness finishes the properness assertion. $\square$

Remark G.4 (The zero-force fibre). For nonconstant q , $Q_q(a) = 0$ forces $a = b_q$, $E_q = 0$, and $\nabla h_a = 0$. The condition $E_q = 0$ reads $\nabla \psi = -b_q \psi_u$. Since ψ has zero vertical mean and is nonconstant, any nonzero Fourier mode (k, m) has $m \neq 0$ and $k + mb_q = 0$. Thus $b_q \in \mathbb{Q}^n$, and q is a rational travelling-wave density. Conversely, such a density preserves the affine foliation at its centre. Therefore a nonuniform prescribed volume has either no zero-force slope or exactly one, at a rational centre.

COROLLARY G.5 (Quadratic source maps are polynomial). *If $K(p) = p^T G p / 2$ with constant $G > 0$, the universal source map $a \mapsto Q_q(a)$ is a polynomial of degree at most two, with coefficients in $C^\infty(M)$. Hence*

$$a \mapsto \int_M |Q_q(a) + V_u|^2 dm$$

is a nonnegative polynomial of degree at most four with zero set exactly $\Sigma_{K,V}(q)$.

Proof. The equation (G.7) is linear in h , has coefficients independent of a , and has right-hand side affine in a . Its unique normalized solution h_a is affine in a . Thus $\widehat{P}_a$ is affine in a , and $R_K(\widehat{P}_a) = \operatorname{div}_x(G\widehat{P}_a) + \partial_u(\widehat{P}_a^T G \widehat{P}_a / 2)$ is quadratic. $\square$

This polynomial representation is an exact test for a fixed prescribed density. The sharper rational-centre geometry of the earlier quadratic theorem uses additional consequences of the full equation, not merely the fact that this polynomial vanishes.

G.3. Analytic zero sets and rigidity tests.

Proof of Theorem F.3. Equation (F.29) gives the analytic zero set. Proposition G.2 gives compactness except in the stated trivial case. In a nontrivial case $\mathcal{F}$ cannot be identically zero: otherwise its spectrum would be all of $\mathbb{R}^n$, contradicting Proposition G.2. A nonzero real-analytic function on the connected set $\mathbb{R}^n$ has a Lebesgue-null zero set. This standard fact also follows

by induction and Fubini: on each coordinate box, outside the zero sets of its nontrivial analytic coefficient functions, the restriction to a coordinate line has isolated zeros.

Fix an affine line $a+th$, $h \neq 0$. The restriction $t \mapsto \mathcal{F}(a+th)$ is real analytic on $\mathbb{R}$. If infinitely many distinct zeros occur, compactness of the spectrum gives a finite accumulation parameter. The identity theorem makes the restriction identically zero, putting an unbounded line in the spectrum. This is impossible.

To obtain a bound uniform over lines, let $S = \Sigma_{K,V}(q)$ be nonempty and proper. For every $(a, h) \in S \times \mathbb{S}^{n-1}$ there is a finite j such that $\partial_h^j \mathcal{F}(a) \neq 0$; otherwise the whole line lies in S . Continuity and a finite covering of this compact product give an integer N and a radius $\rho > 0$ such that, for every (a, h) , some derivative of order at most N remains nonzero along $a+th$ for $|t| < \rho$. By Rolle's theorem, such an interval contains at most N zeros. Enclose S in a ball. Every line intersects the ball in an interval of uniformly bounded length. Partition that interval into subintervals shorter than ρ ; if a subinterval contains a zero, use it as the centre of the preceding derivative estimate. This gives a uniform finite bound. $\square$

Corollary F.4 follows immediately.

Example G.6 (Sharp measure-zero boundary). For $K(p) = |p|^2/2$, choose a nonconstant smooth positive $q = q(u)$ of mean one and $R > 0$, and set $V(u) = -R^2/(2q(u)^2)$. Here $f = 0$, $\nabla_x \psi = 0$, $g_a = 0$, and $P_a = a/q$. A direct calculation gives

$$(G.13) \quad \mathcal{R}(a) = (R^2 - |a|^2) \frac{q'}{q^3}, \quad \mathcal{F}(a) = \left(\int_0^1 \frac{(q')^2}{q^6} du \right) (|a|^2 - R^2)^2.$$

Thus the spectrum is exactly the sphere $|a| = R$. In dimension $n \geq 2$ this gives a nontrivial $(n-1)$ -dimensional spectrum and arbitrarily large finite sets of compatible slopes. It does not provide infinitely many collinear slopes.

Example G.7 (Why smooth kinetic energy is different). Take a nonconstant positive $q = q(u)$ of mean one and a bounded box $S \subset (0, \infty)^n$ separated from the coordinate planes. On a compact interval containing every $a_i/q(u)$ for $a \in S$, choose $\varphi(s) = -\log s$. Extend it to a smooth globally uniformly convex function with bounded Hessian and quadratic tails by prescribing a positive second derivative equal to s^{-2} on that interval and equal to 1 outside a larger interval, and integrating twice. Put

$$K(p) = \sum_i \varphi(p_i), \quad V(u) = n \log q(u), \quad P_a = a/q.$$

On the sampled momenta, $H_K(P_a) = -n + \sum_i \log a_i - n \log q$, hence $R_K(P_a) = -nq'/q = Q$ for every $a \in S$. The phases $\int_0^u q(s)ds - a \cdot x$ give calibrated

minimizing foliations with common volume. Thus a smooth fixed kinetic energy may have a compatibility spectrum with nonempty interior. This is the bounded-set construction of Appendix E, specialized to a positive box. Analyticity cannot be deleted from Theorem F.3.

Appendix H. Finite-jet algebraic compatibility in the wave sector

The analytic selector determines the spectrum for general data, but it is an elliptic rather than a closed-form representation. A rational travelling-wave sector admits a stronger finite algebraic description.

Let $\beta = -k/\ell \in \mathbb{Q}^n$, $\ell > 0$, and set $s = \ell u + k \cdot x$. Suppose

$$(H.1) \quad \begin{aligned} q(x, u) &= r(s), & V(x, u) &= A(x) + F(s), \\ r &> 0, & \int_0^1 r &= 1, & r &\text{nonconstant.} \end{aligned}$$

For $\zeta \in \mathbb{R}^n$ define the kinetic dilation function

$$(H.2) \quad W_\beta(\zeta) = \zeta \cdot DK(\beta + \zeta) - K(\beta + \zeta) + K(\beta).$$

For $j \geq 2$, define its homogeneous Taylor polynomials

$$(H.3) \quad P_j(\zeta) = \frac{j-1}{j!} D^j K(\beta)[\zeta, \dots, \zeta].$$

THEOREM H.1 (Kinetic-jet classification). *Assume (H.1), real-analytic K , and a nonempty compatibility spectrum. Choose $a_0 \in \Sigma_{K,V}(q)$ and $\gamma_0 = a_0 - \beta$. Then*

$$(H.4) \quad F(s) = C - W_\beta(\gamma_0/r(s)).$$

Moreover,

$$(H.5) \quad \Sigma_{K,V}(q) = \beta + \{\gamma \in \mathbb{R}^n : P_j(\gamma) = P_j(\gamma_0) \text{ for every } j \geq 2\}.$$

The set in braces is compact and real algebraic; a finite subset of the polynomial equations suffices. Every compatible field and phase are

$$(H.6) \quad P_a = \beta + \frac{a - \beta}{r(s)}, \quad \tau_a = u - a \cdot x + \frac{1}{\ell} \rho(s), \quad \rho' = r - 1,$$

up to an additive phase constant, with ρ chosen periodic of mean zero.

Conversely, any nonconstant normalized positive r , rational β , and nonzero γ_0 define a nonconstant potential by (H.4); its complete common-volume spectrum is (H.5). No other common-volume fields occur.

Proof. The canonical $\psi = \rho(s)/\ell$ satisfies $\nabla_x \psi = -\beta(q-1)$. Also $f = -\ell \int_0^1 r F'$ is constant in x . Existence forces $f = 0$. In the selector equation the

vertically averaged flux depends on x only through ∇g :

$$\int_0^1 r(s) DK \left(\beta + \frac{a - \beta - \nabla g}{r(s)} \right) ds.$$

The constant function $g = 0$ solves its divergence equation, and uniqueness forces it. Thus the fields are as in (H.6).

Since $k = -\ell\beta$, direct differentiation gives

$$R_K(P_a) = \ell \frac{d}{ds} W_\beta \left(\frac{a - \beta}{r(s)} \right).$$

Therefore compatibility is exactly that $W_\beta(\gamma/r(s)) + F(s)$ is constant. For $\gamma = \gamma_0$ this gives (H.4).

For a second γ , let $t = 1/r(s)$. Its image contains a nondegenerate interval. Compatibility implies $W_\beta(t\gamma) - W_\beta(t\gamma_0)$ is constant on that interval. Both terms are real analytic in $t \in \mathbb{R}$. Their difference is consequently constant everywhere, and evaluation at $t = 0$ makes the constant zero. Its Taylor series at zero is

$$W_\beta(t\gamma) = \sum_{j \geq 2} P_j(\gamma) t^j.$$

Equality is therefore equivalent to all the equations in (H.5). Conversely, equality of the Taylor coefficients gives equality near zero and hence for all real t , so it implies compatibility on the sampled interval. This use of analytic continuation requires no globally convergent Taylor series in p .

The quadratic equation is

$$\gamma^T D^2 K(\beta) \gamma = \gamma_0^T D^2 K(\beta) \gamma_0,$$

so the set is bounded. It is closed. The ideal generated by the polynomial differences $P_j(\gamma) - P_j(\gamma_0)$ in the Noetherian ring $\mathbb{R}[\gamma_1, \dots, \gamma_n]$ is finitely generated; the ascending sequence of ideals generated up to order N stabilizes. Thus finitely many of the displayed equations define the same real set.

Finally, for $t > 0$,

$$\frac{d}{dt} W_\beta(t\gamma_0) = t \gamma_0^T D^2 K(\beta + t\gamma_0) \gamma_0 > 0$$

when $\gamma_0 \neq 0$. Thus the converse potential is nonconstant. It automatically has $\int r F' = 0$, either by direct integration or by the weighted residual identity for its explicit phase. The phase and calibration construction prove existence against all-dimensional competitors. $\square$

Remark H.2. The finite Taylor order need not be uniform over kinetic energies. This is not an effective universal bound on the number of equations. Nor does the theorem assert that an arbitrary nonlinear common-volume family must have the travelling-wave form (H.1).

H.1. *Quadratic and nonlinear examples.* For $K(p) = p^T G p / 2$ with constant $G > 0$, only P_2 survives. The spectrum in (H.5) is the ellipsoid

$$(a - \beta)^T G (a - \beta) = \gamma_0^T G \gamma_0.$$

This recovers the existing quadratic travelling-wave branch as the simplest kinetic-jet case.

Example H.3 (Exactly four slopes for a nonlinear analytic kinetic energy). Let $n = 2$, $0 < \eta < 1$, and

$$(H.7) \quad K(p) = \frac{1}{2}(p_1^2 + p_2^2) + \eta(1 - \cos p_1).$$

Its Hessian is $\text{diag}(1 + \eta \cos p_1, 1)$, so (F.2) holds globally. Choose $\beta = 0$, $\gamma_0 = (A, B)$ with $AB \neq 0$, and any nonconstant positive normalized $r = r(u)$. Define

$$(H.8) \quad V(u) = -W_0(\gamma_0/r(u)), \quad W_0(z) = \frac{1}{2}|z|^2 + \eta(z_1 \sin z_1 + \cos z_1 - 1).$$

Here

$$P_2(\gamma) = \frac{1}{2}((1 + \eta)\gamma_1^2 + \gamma_2^2), \quad P_4(\gamma) = -\frac{\eta}{8}\gamma_1^4.$$

Equality of P_4 implies $\gamma_1^2 = A^2$, and equality of P_2 then implies $\gamma_2^2 = B^2$. Conversely these two conditions make the entire even function $W_0(t\gamma)$ equal to $W_0(t\gamma_0)$. Thus

$$(H.9) \quad \Sigma_{K,V}(r) = \{(A, B), (A, -B), (-A, B), (-A, -B)\}.$$

All four foliations are explicit and class-A minimizing. The spectrum is exactly these four points, not a whole ellipse. In dimension $n > 2$, the corresponding construction gives two $(n - 2)$ -spheres when the perpendicular radius is positive.

H.2. *The rational centre remains a genuine holonomy obstruction.*

PROPOSITION H.4. *For any smooth uniformly convex autonomous kinetic energy and any nonconstant travelling-wave potential $V = A(x) + F(\ell u + k \cdot x)$, no minimizing foliation at $\beta = -k/\ell$ can have identity adjusted rational holonomy.*

Proof. On a rational cover, a closed graph at slope β has $z = \ell u + k \cdot x$ periodic and $\nabla u = \beta + \nabla z/\ell$. After dropping the graph-independent A term and subtracting the energy of a constant maximizing F , convexity gives the cell excess

$$\begin{aligned} & \oint_C \left[K(\beta + \nabla z/\ell) - K(\beta) - DK(\beta) \cdot \nabla z/\ell \right. \\ & \quad \left. + \max F - F(z) \right] \\ & \geq \frac{\lambda}{2\ell^2} \oint_C |\nabla z|^2 + \oint_C (\max F - F(z)). \end{aligned}$$

The omitted affine flux integrates to zero. Constants at maxima attain zero. Thus all periodic global cell minimizers are constants at maxima. Periodic class-A minimizers are global cell minimizers: repeat any improved cell competitor over a large block and join it in a boundary collar; the volume-order improvement dominates the surface-order collar cost. No global upper Hessian bound is needed for this comparison. The given smooth periodic graph, the constant comparison graph, and their fixed-width cutoff interpolations have gradients in one compact set. Smoothness of K therefore bounds the collar energy by $O(R^{n-1})$. Alternatively, a hypothetical closed solution foliation supplies its own calibration, so its leaves must minimize the cell problem directly. Since F is nonconstant, these constant values cannot cover the vertical circle. This excludes every alternative closed minimizing foliation, not merely the phases in (H.6). $\square$

This proposition ties the explicit nonlinear examples to the rational-holonomy issue. They are not counterexamples to the all-rational identity-holonomy implication stated in Section 1: each fails its centre test.

Appendix I. Variable coefficients and a gauge-invariant normal form

A statement about all $L(x, u, p)$ requires a coordinate- and null-divergence-invariant conclusion. In fact it fails as a literal $V_u = 0$ statement already for a scalar variable quadratic kinetic coefficient.

PROPOSITION I.1. *Let $0 < |\varepsilon| < (2\pi)^{-1}$. There are smooth periodic $a > 0$ and V with $V_u \neq 0$ such that*

$$L(x, u, p) = \frac{1}{2}a(x, u)|p|^2 - V(x, u)$$

is uniformly convex in p and has rigid common-volume minimizing foliations at every $\alpha \in \mathbb{R}^n$.

Proof. Write $f(x) = \cos(2\pi x_1)$, $g(u) = \cos(2\pi u)$, and $G(u) = \sin(2\pi u)/(2\pi)$. Set

$$\Phi(x, u) = u + \varepsilon f(x)g(u), \quad a = \Phi_u^2,$$

and introduce the periodic vector field

$$F(x, u) = \varepsilon \nabla f(x)G(u) + \frac{\varepsilon^2}{2}f(x)\nabla f(x)g(u)^2.$$

Then $F_u = \Phi_u \nabla_x \Phi$. Put

$$V = \operatorname{div}_x F - \frac{1}{2}|\nabla_x \Phi|^2 = -2\pi\varepsilon f(x)\sin(2\pi u) - 2\pi^2\varepsilon^2 f(x)^2 \cos^2(2\pi u).$$

For every graph u the exact identity is

$$(I.1) \quad L(x, u, \nabla u) = \frac{1}{2} |\nabla_x [\Phi(x, u(x))]|^2 - \operatorname{div}_x [F(x, u(x))].$$

The bound on ε makes $\Phi_u > 0$ and a uniformly positive. The fibre map $u \mapsto \Phi(x, u)$ is a degree-one diffeomorphism, so the pullbacks of affine graphs

$$\Phi(x, U_\alpha(x, t)) = \alpha \cdot x + t$$

form smooth rigid foliations at every slope. They minimize against every fixed-trace competitor by (I.1), because the flat Dirichlet energy is minimized by affine graphs. The inverse phase is $\tau_\alpha = \Phi - \alpha \cdot x$, and hence every foliation has the same invariant probability density $q = \Phi_u$. Finally,

$$V_u(0, 0) = -4\pi^2\varepsilon \neq 0.$$

□

This example is not in the autonomous kinetic class (E.1). It does not disprove the original fixed Euclidean holonomy implication. It does show that even an all-slope common-volume theorem for variable coefficients must classify an equivalence class rather than force the displayed potential to be independent of u in every coordinate gauge. The role of null divergences is classical; see [10] for the broader variational setting.

I.1. A complete affine-graph normal form. The following elementary classification gives a precise normal-form target. Its global momentum-domain hypothesis is essential.

THEOREM I.2 (Affine extremals modulo null divergences). *Let $L \in C^\infty(\mathbb{T}^n \times \mathbb{T} \times \mathbb{R}^n)$. Every affine graph $u(x) = p \cdot x + c$, for every $p \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, is an Euler–Lagrange solution if and only if*

$$(I.2) \quad L(x, u, p) = L_0(x, p) + \operatorname{div}_x F(x, u) + F_u(x, u) \cdot p,$$

where F is smooth and periodic, L_0 is independent of u , and

$$(I.3) \quad \operatorname{div}_x D_p L_0(x, p) = 0 \quad \text{for every fixed } p.$$

One may take $L_0(x, p) = \int_0^1 L(x, u, p) du$ and normalize F to have zero vertical mean. If L is convex in p , these affine extremals are class-A minimizers.

Proof. On an affine graph the Euler–Lagrange equation is exactly

$$(I.4) \quad \sum_i L_{x_i p_i} + \sum_i p_i L_{u p_i} - L_u = 0.$$

Expand L in Fourier series in (x, u) , writing $\ell_{k,m}(p)$ for a coefficient. For $m \neq 0$, (I.4) becomes

$$(k + mp) \cdot \nabla_p \ell_{k,m} = m \ell_{k,m}.$$

With $p_* = -k/m$ this is

$$(p - p_*) \cdot \nabla_p \ell_{k,m} = \ell_{k,m}.$$

Along each ray from p_* the coefficient is positively homogeneous of degree one. Smoothness at p_* makes it linear:

$$\ell_{k,m}(p) = c_{k,m} \cdot (p - p_*).$$

For example, $\ell(p_* + tv) = t\ell(p_* + v)$ for $t > 0$; differentiating at $t = 0$ gives $\ell(p_* + v) = D\ell(p_*) \cdot v$. Thus $W = L - L_0$ is affine in p , say

$$W = A(x, u) \cdot p + B(x, u),$$

with A, B smooth and of zero vertical mean. Averaging (I.4) gives (I.3); subtracting it gives $\operatorname{div}_x A = B_u$. Let F be the zero-mean periodic primitive with $F_u = A$. Then $\partial_u(\operatorname{div}_x F - B) = 0$ and its vertical mean is zero, so $\operatorname{div}_x F = B$. This proves (I.2).

Conversely the last two terms in (I.2) are the total divergence of $F(x, u(x))$, so do not change the Euler equation. Condition (I.3) makes all affine graphs extremals for L_0 . When L is convex in p , so is L_0 . For $u = p \cdot x + c$ and a fixed-trace competitor $u + \eta$, convexity gives

$$\int [L_0(x, p + \nabla \eta) - L_0(x, p)] dx \geq \int D_p L_0(x, p) \cdot \nabla \eta dx = 0.$$

The null-divergence integrals agree, giving minimality. $\square$

The Fourier argument uses smoothness at every resonant point $p_* = -k/m$. An assertion only on a bounded open momentum set cannot invoke those points when they lie outside the set. This is consistent with the logarithmic-patch counterexamples.

COROLLARY I.3 (A common affine phase coordinate). *Suppose a periodic scalar integrand has solution foliations for every slope whose inverse phases are*

$$\tau_\alpha(x, u) = \Phi(x, u) - \alpha \cdot x, \quad \Phi_u > 0, \quad \Phi(x, u) - u \text{ periodic.}$$

Let $u = U(x, v)$ invert $v = \Phi(x, u)$, and define

$$\tilde{L}(x, v, r) = L(x, U(x, v), \nabla_x U(x, v) + U_v(x, v)r).$$

Then $\tilde{L}$ has the normal form (I.2)–(I.3), with v, r in place of u, p .

Proof. In the new vertical coordinate every selected leaf is $v = \alpha \cdot x + t$. The invertible point transformation preserves the Euler–Lagrange extremal property. Apply Theorem I.2. $\square$

Common volume alone is not being substituted for the affine phase-coordinate hypothesis of this corollary. For general kinetics, the corrector

equation (E.17) is one obstruction to making that substitution. Nor must L_0 be independent of x : in dimension at least two,

$$L_0(x, p) = \frac{1}{2}|p|^2 + \frac{1}{2}a(x_1)p_2^2, \quad a > 0 \text{ periodic,}$$

satisfies (I.3) with nonconstant a .

Appendix J. Fixed-trace comparisons and Sobolev continuity

This appendix supplies the approximation details used in the calibration arguments. They are independent of the slope and hold in every dimension.

LEMMA J.1. *Let Ω be bounded and Lipschitz, and let $v_j \rightarrow v$ in $W^{1,2}(\Omega)$. Then*

$$\mathcal{A}_V[v_j; \Omega] \longrightarrow \mathcal{A}_V[v; \Omega].$$

Consequently, class-A comparison against smooth compactly supported variations is equivalent to comparison against all $W_0^{1,2}$ variations.

Proof. For the kinetic term,

$$\left| \int_{\Omega} (|\nabla v_j|^2 - |\nabla v|^2) dx \right| \leq \|\nabla v_j - \nabla v\|_2 (\|\nabla v_j\|_2 + \|\nabla v\|_2) \rightarrow 0.$$

Periodicity makes V_u bounded, so

$$\int_{\Omega} |V(x, v_j) - V(x, v)| dx \leq \|V_u\|_{\infty} |\Omega|^{1/2} \|v_j - v\|_2 \rightarrow 0.$$

If $w - v \in W_0^{1,2}(\Omega)$, choose $\eta_j \in C_c^{\infty}(\Omega)$ converging to $w - v$ in $W^{1,2}$. Apply the smooth-variation inequality to $v + \eta_j$ and pass to the limit. The converse is immediate. $\square$

For $p \in C^{\infty}(M; \mathbb{R}^n)$, the functional

$$N[w] = \int_{\Omega} \left(p(x, w) \cdot \nabla w - \frac{1}{2}|p(x, w)|^2 - V(x, w) \right) dx$$

is also continuous under strong $W^{1,2}$ convergence. Indeed, the only term requiring more than bounded Lipschitz continuity is the product with the gradient. Its difference can be written as

$$p(x, w_j) \cdot (\nabla w_j - \nabla w) + (p(x, w_j) - p(x, w)) \cdot \nabla w.$$

The first integral is bounded in absolute value by $\|p\|_{\infty} |\Omega|^{1/2} \|\nabla w_j - \nabla w\|_2$, and the second by $\|p_u\|_{\infty} \|w_j - w\|_2 \|\nabla w\|_2$. Therefore the null-variation calculation in Lemma 4.5 extends from smooth compactly supported differences to every fixed-trace competitor. The right-hand side of (4.10) converges as well, since $p(x, w_j) \rightarrow p(x, w)$ in L^2 .

For the autonomous primitive in Lemma 2.4, a globally Lipschitz G_E satisfies the Sobolev chain rule and commutes with the trace operator. Thus

$G_E(w) - G_E(v) \in W_0^{1,2}(\Omega)$ whenever $w - v \in W_0^{1,2}(\Omega)$. Integrating the weak derivative of such a function over Ω gives zero, by approximation with compactly supported smooth functions. This justifies the divergence cancellation in (2.9) for arbitrary Sobolev competitors.

These arguments impose no boundedness condition on the competing Sobolev function. Boundedness of the periodic coefficients, rather than boundedness of w , is the relevant fact.

Appendix K. Fixed-cell response and exact amplitude isolation

For a rational slope a , choose $L \in \mathbb{N}$ with $La \in \mathbb{Z}^n$ and write $C = \mathbb{R}^n / L\mathbb{Z}^n$. Throughout this appendix the canonical mean normalization is

$$(K.1) \quad U_c(x) = a \cdot x + c + w_c(x), \quad \langle w_c \rangle_C = 0, \quad \langle h \rangle_C = |C|^{-1} \int_C h \, dx.$$

The statements here concern a coupling parameter on a fixed rational cell, not differentiation in slope. Write $Q = -V_u$ and

$$E_{\lambda V}[v] = \int_C \left(\frac{1}{2} |\nabla v|^2 - \lambda V(x, v) \right) dx.$$

The equation is $\Delta v = \lambda Q(x, v)$. This appendix proves an exact two-coupling example and the analytic isolated-parameter alternative. These are independent of the common-volume closure.

K.1. Two fixed-cell facts.

LEMMA K.1 (Sliding uniqueness). *Suppose $\Delta u = \lambda Q(x, u)$ admits a closed ordered solution foliation of slope a on C . Every other solution with the same affine increments is a leaf of this foliation. In particular, there is exactly one such solution of each mean height c .*

Proof. Let $\tau(x, u)$ be the lifted inverse phase of the foliation and let v be the other solution. The function $x \mapsto \tau(x, v(x))$ is periodic on C : the affine deck increments of the graph cancel those of the inverse phase. Let c_* be its maximum. Then $d = U_{c_*} - v \geq 0$ and d vanishes somewhere. Subtracting the two equations gives

$$\Delta d = A(x)d, \quad A(x) = \lambda \int_0^1 Q_u(x, v + s(U_{c_*} - v)) \, ds.$$

The coefficient is bounded. The Harnack inequality for nonnegative solutions, or the strong maximum principle in its Schrödinger form, implies $d = 0$. The mean normalization identifies the leaf uniquely. $\square$

LEMMA K.2 (Calibration and cell comparison). *A smooth closed solution foliation calibrates all its leaves against arbitrary fixed-trace competitors. Its leaves also minimize the normalized action among all affine-periodic graphs with*

the same slope. A periodic class-A minimizer is a minimizer of the corresponding cell action.

Proof. Let $p(x, u)$ be the gradient of the leaf through (x, u) . It is periodic on the rational cover. The leaf equation reads

$$\operatorname{div}_x p + \partial_u \left(\frac{1}{2} |p|^2 + \lambda V \right) = 0.$$

With $dx = dx_1 \wedge \cdots \wedge dx_n$ and $dx_i^\# = \iota_{\partial_{x_i}} dx$, the form

$$\omega_p = \sum_i p_i du \wedge dx_i^\# - \left(\frac{1}{2} |p|^2 + \lambda V \right) dx$$

is closed. For every graph v ,

$$(K.2) \quad \left(\frac{1}{2} |\nabla v|^2 - \lambda V(x, v) \right) dx = v^* \omega_p + \frac{1}{2} |\nabla v - p(x, v)|^2 dx.$$

The integral of the closed form agrees on fixed-trace graphs, by Stokes applied to the vertical graph homotopy. It also agrees on affine-periodic graphs of the same slope, which are homologous on the rational quotient. The square vanishes on a leaf. This proves the first two claims. Smooth approximation gives the fixed-trace comparison for $W^{1,2}$ competitors because V and p are bounded on their periodic domains.

For the last claim, a cell competitor with strictly smaller average action can be tiled over a large number of cells, then joined to the original graph in a collar of fixed width. The periodic difference of the graphs and their gradients is bounded. The negative interior gain is of order R^n while the collar cost is of order R^{n-1} . For sufficiently large R this contradicts class-A minimality. $\square$

A foliation first constructed only on a finite cover is invariant under unit deck translations. Indeed, translating any leaf by an integer vector produces a solution with the same affine increments, so Lemma K.1 puts it in the family. The mean label changes by $a \cdot m$ under $x \mapsto x + m$, since the corrector's integral is translation invariant. Thus the mean label gives the rigid deck law

$$(K.3) \quad U(x + m, c) = U(x, c + a \cdot m).$$

This also verifies that the fixed-cell formulation agrees with the required rational holonomy condition.

K.2. *An exact two-coupling counterexample to continuation.*

THEOREM K.3 (Exactly two foliation couplings). *For every $n \geq 1$ and every $0 < |\epsilon| < 1/(2\pi)$, there is a nonconstant real-analytic V_ϵ on M such that*

$$(K.4) \quad \{\lambda \in \mathbb{R} : \lambda V_\epsilon \text{ has a closed zero-slope minimizing foliation}\} = \{0, 1\}.$$

The same assertion holds when closure is allowed on any full-rank finite cover of the base torus. At all other couplings, there is not even a smooth closed ordered solution foliation.

K.2.1. *The potential and the foliation at coupling one.* Set

$$(K.5) \quad f(x) = \cos(2\pi x_1), \quad g(c) = \cos(2\pi c), \quad U_\epsilon(x, c) = c + \epsilon f(x)g(c).$$

Then

$$\phi_\epsilon = \partial_c U_\epsilon = 1 - 2\pi\epsilon f(x) \sin(2\pi c) > 0.$$

The mean of U_ϵ is c , and $U_\epsilon(x, c+1) = U_\epsilon(x, c) + 1$. Let $T_\epsilon(x, u)$ be the inverse in c . It is real analytic, with the corresponding periodicity. Define V_ϵ by the explicit pullback formula

$$(K.6) \quad V_\epsilon(x, U_\epsilon(x, c)) = 2\pi\epsilon f(x) \sin(2\pi c) + 2\pi^2\epsilon^2 f(x)^2 \cos^2(2\pi c).$$

The right side is periodic in x, c , so this defines a real-analytic periodic physical potential. Differentiation gives

$$\begin{aligned} \partial_c[V_\epsilon(x, U_\epsilon)] &= 4\pi^2\epsilon f \cos(2\pi c) - 8\pi^3\epsilon^2 f^2 \cos(2\pi c) \sin(2\pi c) \\ &= 4\pi^2\epsilon f \cos(2\pi c) \phi_\epsilon. \end{aligned}$$

Consequently

$$(K.7) \quad Q_\epsilon(x, U_\epsilon) := -V_{\epsilon, u}(x, U_\epsilon) = -4\pi^2\epsilon f(x) \cos(2\pi c) = \Delta_x U_\epsilon.$$

Thus (K.5) is a smooth closed solution foliation at $\lambda = 1$. It is class-A minimizing by Lemma K.2, including against variations depending on all n base variables. The source is nonzero, for example at $x_1 = 0, c = 0$.

Write the normalized zero-slope cell action as

$$E_\lambda[v] = \int_{\mathbb{T}^n} \left(\frac{1}{2} |\nabla v|^2 - \lambda V_\epsilon(x, v) \right) dx.$$

Since $\int f = 0$, $\int f^2 = 1/2$, and $\int |\nabla f|^2 = 2\pi^2$,

$$(K.8) \quad E_0[U_\epsilon(\cdot, c)] = \pi^2\epsilon^2 g(c)^2, \quad \int_{\mathbb{T}^n} V_\epsilon(x, U_\epsilon) = \pi^2\epsilon^2 g(c)^2, \quad E_1[U_\epsilon(\cdot, c)] = 0.$$

Calibration therefore gives $E_1[v] \geq 0$ for every periodic graph v .

K.2.2. *The two constant leaves.* The values

$$c_+ = \frac{1}{4}, \quad c_- = \frac{3}{4}$$

satisfy $g(c_\pm) = 0$. They give constant graphs and

$$(K.9) \quad Q_\epsilon(x, c_\pm) = 0, \quad V_\epsilon(x, c_+) = 2\pi\epsilon f(x), \quad V_\epsilon(x, c_-) = -2\pi\epsilon f(x).$$

Hence they are classical solutions for every λ , with $E_\lambda[c_\pm] = 0$.

K.2.3. *Exclusion of the entire interval between zero and one.* For $0 < \lambda < 1$,

$$(K.10) \quad E_\lambda[v] = (1 - \lambda)E_0[v] + \lambda E_1[v] \geq 0.$$

The two constant leaves attain zero. A cell minimizer must therefore have $E_0[v] = E_1[v] = 0$. The first equality makes v constant. The second puts it in the foliation at coupling one, by the equality case of the calibration (or Lemma K.1). The only constant leaves of (K.5) are $c_\pm$ modulo integers. Thus these are the only periodic cell minimizers, and they cannot fill the vertical circle. Every closed minimizing foliation is excluded.

K.2.4. *Exclusion of couplings greater than one.* For $\lambda > 1$, take the leaf with mean $c = 0$ as a competitor. Equation (K.8) gives

$$E_\lambda[U_\epsilon(\cdot, 0)] = (1 - \lambda)\pi^2\epsilon^2 < 0.$$

The constant classical solution c_+ has action zero and is not a cell minimizer. If a closed solution foliation existed, Lemma K.1 would make c_+ one of its leaves, and Lemma K.2 would make it a cell minimizer. This contradiction excludes the foliation.

K.2.5. *Exclusion of negative couplings.* Compute the physical Haar average using $du = \phi_\epsilon dc$. Direct integration of (K.6) gives

$$(K.11) \quad \int_M V_\epsilon(x, u) dx du = -\pi^2\epsilon^2 \int_{\mathbb{T}^n} f(x)^2 dx = -\frac{\pi^2\epsilon^2}{2} < 0.$$

In detail, the two quadratic contributions after the c integration are $-2\pi^2\epsilon^2 f^2$ and $+\pi^2\epsilon^2 f^2$; the linear and cubic terms integrate to zero. Therefore some constant graph $u = d$ has $\int V_\epsilon(x, d) dx < 0$. If $\lambda < 0$, then $E_\lambda[d] < 0$, whereas $E_\lambda[c_+] = 0$. Again c_+ is a classical solution that cannot be a leaf of any minimizing closed foliation. Sliding and calibration give the same contradiction.

At $\lambda = 0$, the constant affine family is a minimizing foliation. At $\lambda = 1$, (K.5) supplies one. This proves (K.4). All comparisons hold on every finite cover after normalization by its volume: the calibration has zero period on zero-slope graphs, and all explicit averages are averages of unit-periodic functions. This proves the remaining assertions of Theorem K.3.

COROLLARY K.4 (A prescribed rational test). *For any $a = -k/\ell \in \mathbb{Q}^n$, the potential*

$$\widetilde{V}_\epsilon(x, u) = \ell^{-2}V_\epsilon(x, \ell u + k \cdot x)$$

has a closed minimizing foliation of slope a at exactly the couplings $\lambda = 0, 1$.

Proof. Put $v = \ell u + k \cdot x$. Then

$$\mathcal{A}_{\lambda \widetilde{V}_\epsilon}[u; \Omega] = \ell^{-2} \mathcal{A}_{\lambda V_\epsilon}[v; \Omega] - \ell^{-2} \int_\Omega k \cdot \nabla v dx + \frac{|k|^2}{2\ell^2} |\Omega|.$$

The last two terms are fixed for fixed-trace or same-slope cell competitors. At slope a , closed u corresponds exactly to periodic v on a finite cover. Theorem K.3 applies. At coupling one the lifted foliation is $(U_\epsilon(x, \ell c) - k \cdot x)/\ell$; its vertical degree and rational closure are immediate. $\square$

K.3. *The actual fixed-cell constrained response.* We now return to an arbitrary smooth periodic V . Fix a, C and a coupling λ_0 with a closed foliation. Work in the mean gauge (K.1). The periodic Jacobi operator is

$$(K.12) \quad L_c = -\Delta + \lambda_0 Q_u(x, U_c), \quad L_c \phi_c = 0.$$

The sign is important: $Q_u = -V_{uu}$, so this is the second-variation operator. For every periodic η ,

$$(K.13) \quad \langle |\nabla \eta|^2 + \lambda_0 Q_u(x, U_c) \eta^2 \rangle_C = \langle \phi_c^2 |\nabla(\eta/\phi_c)|^2 \rangle_C.$$

Thus L_c is nonnegative and its periodic kernel is exactly $\mathbb{R}\phi_c$.

Choose $0 < \gamma < 1$ and let $X = C_{\text{per},0}^{2,\gamma}(C)$, the mean-zero space. The correct bordered equation is

$$(K.14) \quad \mathcal{G}(\lambda, c, w, \mu) = -\Delta w + \lambda Q(x, a \cdot x + c + w) - \mu = 0, \quad (w, \mu) \in X \times \mathbb{R}.$$

The multiplier μ is retained, not set equal to zero by a phase-flatness assertion.

PROPOSITION K.5 (Bordered invertibility and response). *At a foliated coupling, the derivative in (w, μ) of (K.14) is an isomorphism from $X \times \mathbb{R}$ to $C_{\text{per}}^{0,\gamma}(C)$. The implicit-function theorem produces $w(\lambda, c)$ and $\mu(\lambda, c)$ locally, smoothly for smooth V and real analytically for real-analytic V . Moreover,*

$$(K.15) \quad \partial_\lambda \mu(\lambda_0, c) = \langle Q(x, U_c) \phi_c \rangle_C = -\frac{d}{dc} \langle V(x, U_c) \rangle_C.$$

Proof. The derivative is $(\eta, \nu) \mapsto L_c \eta - \nu$. To solve $L_c \eta - \nu = h$, pair with ϕ_c and use $\langle \phi_c \rangle_C = 1$, obtaining $\nu = -\langle \phi_c h \rangle_C$. Then $h + \nu$ is orthogonal to the kernel of the self-adjoint operator L_c , so the Fredholm alternative supplies a solution of $L_c \eta = h + \nu$. Adding a unique multiple of ϕ_c enforces $\langle \eta \rangle_C = 0$. The elliptic estimates give a bounded inverse. Compactness in $c \bmod 1$ provides a common local parameter neighborhood.

Differentiate (K.14) in λ at the reference foliation:

$$L_c w_\lambda + Q(x, U_c) - \mu_\lambda = 0.$$

Pairing with ϕ_c proves the first equality of (K.15). The second is the chain rule and $Q = -V_u$. Analyticity follows from the analytic composition map in Hölder spaces and the analytic implicit-function theorem; real-analytic periodic Q has a common complex neighborhood of the compact physical torus. $\square$

For the explicit construction, (K.8) yields the exact formula

$$(K.16) \quad \mu_\lambda(1, c) = -2\pi^2 \epsilon^2 g(c) g'(c) = 2\pi^3 \epsilon^2 \sin(4\pi c).$$

In particular the response obstruction is nonzero at $c = 1/8$. The global variational proof of Theorem K.3 is stronger than this local derivative test: it determines all allowable real couplings, not just a punctured neighborhood of one.

K.4. *Compactness and capture on one rational cover.* The preceding implicit-function theorem describes the local constrained branch. To identify it with all nearby foliations requires a separate compactness argument.

LEMMA K.6 (Fixed-cell compactness). *For λ in a compact interval and c in a compact interval, all affine-periodic solutions with mean c have uniform spatial C^j bounds on their mean-zero correctors, for every finite j . If these solutions form ordered foliations, the mean-normalized positive transverse derivatives have uniform positive upper and lower bounds on $C \times [0, 1]$. The set of foliated couplings is closed.*

Proof. The equation

$$\Delta w = \lambda Q(x, a \cdot x + c + w), \quad \langle w \rangle_C = 0$$

has uniformly bounded right-hand side. The inverse periodic Laplacian and $W^{2,p}$ estimates, with $p > n$, give a uniform $C^{1,\gamma}$ bound. Substitution into the smooth right-hand side and elliptic bootstrapping give all spatial bounds. The cell and slope are fixed throughout.

For $h > 0$, let $z_h = (U_{c+h} - U_c)/h$. Then $z_h > 0$, $\langle z_h \rangle_C = 1$, and

$$\Delta z_h = A_h(x, c)z_h, \quad A_h = \lambda \int_0^1 Q_u(x, U_c + s(U_{c+h} - U_c)) ds.$$

The coefficient is uniformly bounded. Global Harnack on the fixed compact cell and the mean normalization give $0 < m \leq z_h \leq M$. The spatial bounds on the coefficients and elliptic estimates give compactness of the difference quotients. Any limit solves $L_c z = 0$ and has mean one. A positive kernel element and the ground-state identity make this normalized limit unique. Thus $\partial_c U_c$ exists, is positive, and is the unique normalized periodic ground state. The bordered inverse, applied locally to this existing family at fixed coupling, gives smooth transverse dependence; for analytic V it gives analytic dependence. This proves the regularity needed without assuming it in a limiting foliation.

For a sequence of foliated couplings converging to λ_* , the estimates and Arzelà–Ascoli give a subsequence of mean-normalized families converging uniformly in $c \in [0, 1]$ and locally in the spatial C^j topologies. The lower difference-quotient bound preserves strict ordering; the degree-one identity preserves filling. Passing to the equation gives a closed solution foliation at λ_* . The preceding transverse argument regularizes it, and calibration makes it minimizing. Thus the foliation set is closed. $\square$

LEMMA K.7 (Capture). *In a neighborhood of a foliated coupling λ_0 , closed solution foliations exist precisely when the constrained branch of (K.14) satisfies $\mu(\lambda, c) = 0$ for every $c \bmod 1$.*

Proof. If a sequence of actual solutions at nearby couplings, with prescribed means, escaped the local constrained branch, the fixed-cell bounds would give a convergent subsequence. Its limit is a solution at λ_0 , and sliding uniqueness makes it the reference leaf of the limiting mean. This contradicts the local uniqueness in the bordered implicit-function theorem. Compactness of the phase circle makes capture uniform in c .

Conversely, if $\mu = 0$ at every phase, the constrained branch is an actual solution family. Its transverse derivative remains positive by continuity from the reference compact family, and its degree-one property follows from uniqueness and the periodicity of Q . It is therefore a closed foliation, and calibration applies. Unit deck invariance follows from sliding as in (K.3). $\square$

K.5. *Analytic amplitude isolation at arbitrary coupling.* For a fixed rational slope define

$$\mathcal{F}_a(V) = \{\lambda \in \mathbb{R} : \lambda V \text{ admits a closed solution foliation of slope } a\}.$$

The definition is independent of taking a larger rational cell. Indeed, if a foliation exists on a larger cover, translating by the smaller rational deck vectors and applying sliding uniqueness yields the required smaller-cell closure.

THEOREM K.8 (Fixed-slope analytic dichotomy). *Let V be real analytic and periodic. For each rational a , the set $\mathcal{F}_a(V)$ is either all of $\mathbb{R}$ or a closed discrete subset of $\mathbb{R}$. If*

$$(K.17) \quad H_a(c) := \langle Q(x, a \cdot x + c) \rangle_C \neq 0,$$

then the second alternative holds. A closed discrete set here is finite on every bounded interval; no global finiteness claim is made.

Proof. The set is closed by Lemma K.6. Suppose distinct $\lambda_j \in \mathcal{F}_a(V)$ converge to λ_* . Closedness puts λ_* in the set. Construct the analytic bordered branch near λ_* . Capture gives $\mu(\lambda_j, c) = 0$ for every c . For each fixed c , the real-analytic function $\lambda \mapsto \mu(\lambda, c)$ has accumulating zeros and therefore vanishes on the common local interval. Lemma K.7 puts that interval in $\mathcal{F}_a(V)$.

An interval contained in $\mathcal{F}_a(V)$ cannot have a finite terminal endpoint. Such an endpoint belongs to the closed set and is again an accumulation point; the same argument extends the interval past it. Thus the existence of any finite accumulation point implies $\mathcal{F}_a(V) = \mathbb{R}$. Otherwise the set is closed and discrete.

At $\lambda = 0$, the mean-normalized family is $U_c = a \cdot x + c$, with $\phi_c = 1$. Formula (K.15) gives $\mu_\lambda(0, c) = H_a(c)$. If (K.17) holds, some phase has a

nonzero first derivative, so zero is an isolated foliated coupling. The whole-line alternative is impossible. $\square$

COROLLARY K.9 (All-rational coupling set). *For a nonzero real-analytic source $Q = -V_u$, the set*

$$\mathcal{I}(V) = \{\lambda \in \mathbb{R} : \lambda V \text{ satisfies } \text{RH}_{\mathbb{Q}}\}$$

is closed and discrete. Consequently, if distinct all-rational foliation couplings have a finite accumulation point, then $V_u = 0$.

Proof. Choose $\widehat{Q}(k, \ell) \neq 0$ with $\ell \neq 0$. Such a coefficient exists because the vertical mean of Q is zero. Replacing (k, ℓ) by its negative allows $\ell > 0$. Set $a = -k/\ell$ and use the cell $[0, \ell]^n$. A change of variable $u = a \cdot x + c$ gives

$$\begin{aligned} \int_0^1 H_a(c) e^{-2\pi i \ell c} dc &= \left\langle e^{-2\pi i k \cdot x} \int_0^1 Q(x, u) e^{-2\pi i \ell u} du \right\rangle_C \\ &= \widehat{Q}(k, \ell) \neq 0. \end{aligned}$$

The averaging in the last line is over an integer number of unit cells. Theorem K.8 makes this $\mathcal{F}_a(V)$ discrete. The intersection $\mathcal{I}(V) = \bigcap_{a \in \mathbb{Q}^n} \mathcal{F}_a(V)$ is closed and contained in that discrete set. $\square$

Remark K.10 (The precise unresolved alternative). Corollary K.9 does not prove $\mathcal{I}(V) = \{0\}$. In particular, it does not exclude an isolated point $\lambda = 1$ in the all-rational intersection. The exact two-point example proves that an isolated nonzero point cannot be excluded using a single rational slope. It does not prove that such a point exists simultaneously for all rational slopes. Analyticity is an additional hypothesis for this section; it has not been derived from the original smooth holonomy hypothesis.

K.6. *The example fails another rational test.*

PROPOSITION K.11 (An explicit missing slope in the example). *For $0 < |\epsilon| \leq 10^{-3}$, the potential in Theorem K.3 has no closed minimizing foliation of slope e_1 at coupling one. In particular, this example does not disprove the all-rational implication.*

Proof. Write $\delta = |\epsilon|$. If such a foliation existed, its leaves would be unit affine-periodic, by sliding and the integer slope. Since the potential depends only on x_1, u , the zero-mean transverse derivatives U_{x_j} , $j \geq 2$, lie in the simple positive periodic Jacobi kernel and must vanish, because a nonzero zero-mean function cannot belong to a one-dimensional kernel generated by a positive function. It is enough to consider the one-dimensional mean-gauged leaves $U_c(x) = x + c + w_c(x)$, with w_c periodic and mean zero.

The construction gives the global bounds

$$\|Q_\epsilon\|_\infty \leq 4\pi^2\delta, \quad \|Q_{\epsilon,u}\|_\infty \leq \frac{8\pi^3\delta}{1-2\pi\delta}.$$

The mean-zero inverse of the one-dimensional periodic Laplacian has kernel of absolute value at most $1/12$. Hence

$$\|w_c\|_\infty \leq \frac{1}{12}\|Q_\epsilon\|_\infty.$$

Averaging the leaf equation and comparing it with the affine graph yields

$$\left| \int_0^1 Q_\epsilon(x, x+c) dx \right| \leq \frac{1}{12}\|Q_\epsilon\|_\infty\|Q_{\epsilon,u}\|_\infty.$$

The first Fourier coefficient in c is $\widehat{Q}_\epsilon(-1, 1)$, so

$$(K.18) \quad |\widehat{Q}_\epsilon(-1, 1)| \leq \frac{8\pi^5\delta^2}{3(1-2\pi\delta)}.$$

On the other hand, the inverse identity gives $|T_\epsilon(x, u) - u| \leq \delta$. Therefore

$$|Q_\epsilon(x, u) + 4\pi^2\epsilon \cos(2\pi x) \cos(2\pi u)| \leq 8\pi^3\delta^2.$$

Taking the same Fourier coefficient gives

$$(K.19) \quad |\widehat{Q}_\epsilon(-1, 1)| \geq \pi^2\delta - 8\pi^3\delta^2.$$

Compatibility of (K.18) and (K.19) would require

$$(1 - 8\pi\delta)(1 - 2\pi\delta) \leq \frac{8}{3}\pi^3\delta.$$

For $0 < \delta \leq 10^{-3}$, using $\pi < 22/7$, the left side is greater than 0.968 and the right side is less than 0.083. This is a contradiction. $\square$

Appendix L. Affine symmetries and exact Fourier-rank reduction

L.1. *Affine symmetry localizes every rational obstruction.* An additive function of x changes neither the Euler equation nor any minimizing comparison. We therefore replace V by its zero-vertical-mean representative

$$(L.1) \quad V^0(x, u) = V(x, u) - \int_0^1 V(x, s) ds.$$

Let its continuous affine symmetry space be

$$(L.2) \quad \mathcal{S} = \{(a, b) \in \mathbb{R}^n \times \mathbb{R} : a \cdot \nabla_x V^0 + bV_u^0 = 0\}.$$

The same space is obtained from Q . Indeed $D_{(a,b)}Q = 0$ makes $D_{(a,b)}V^0$ independent of u , and its vertical mean is zero. Hence that derivative vanishes.

THEOREM L.1 (Symmetry dichotomy). *Let $(a, b) \in \mathcal{S}$, and let $\alpha \in \mathbb{Q}^n$.*

- (i) If $a \cdot \alpha \neq b$, a smooth closed class-A minimizing foliation exists at slope α , without any further condition on the potential.
 - (ii) If $a \cdot \alpha = b$, every global periodic cell minimizer at that slope satisfies
- (L.3)
$$a \cdot \nabla U = b.$$

Proof. Take a global cell minimum U from Lemma A.1. Define

$$h = a \cdot \nabla U - b, \quad d = \langle h \rangle_C = a \cdot \alpha - b.$$

Differentiation of the equation and the symmetry of Q give

$$\Delta h = a \cdot Q_x(x, U) + Q_u(x, U)a \cdot \nabla U = Q_u(x, U)h.$$

Thus $\mathcal{L}_U h = 0$. Since $\mathcal{L}_U \geq 0$, any nonzero kernel is its simple positive ground eigenspace. A nonzero h has one strict sign. If $d = 0$ this is impossible, so $h = 0$, proving the second assertion. Notice that no pre-existing foliation is needed for this conclusion.

If $d \neq 0$, then $h/d > 0$. Define

$$U_t(x) = U(x + ta) - td.$$

Symmetry makes these solutions and preserves their cell energies. The mean height of U_t is $c_0 + td$, because translating the periodic corrector preserves its average. Parameterize by $c = c_0 + td$. Then

$$\partial_c U_c(x) = \frac{h(x + ta)}{d} > 0.$$

The bounded periodic corrector and the term td show that the family fills each vertical fibre as c ranges over $\mathbb{R}$. It has fixed affine increments on C . Sliding and mean normalization give (A.4), so it is a natural closed foliation on the original torus. Calibration gives class-A minimality against all-dimensional competitors. This proves the first assertion. $\square$

COROLLARY L.2 (The only potentially obstructed slopes). *Define*

$$(L.4) \quad K = \{\alpha \in \mathbb{R}^n : a \cdot \alpha = b \text{ for every } (a, b) \in \mathcal{S}\}.$$

Every rational slope outside K has a closed minimizing foliation. Every global periodic minimum at a rational slope in K inherits every symmetry in $\mathcal{S}$.

L.2. *Fourier rank and an exact lower-dimensional quotient.* Assume $Q \neq 0$. Its nonzero Fourier coefficients have vertical frequency $m \neq 0$. Define

$$(L.5) \quad \mathcal{L} = \text{span}_{\mathbb{R}}\{(k, m) \in \mathbb{Z}^n \times \mathbb{Z} : \widehat{Q}(k, m) \neq 0\}, \quad r = \dim \mathcal{L}.$$

This is a rational subspace and $1 \leq r \leq n + 1$; no finiteness of Fourier support is assumed. Fourier uniqueness gives $\mathcal{S} = \mathcal{L}^\perp$ and the direct description

$$K = \{\alpha \in \mathbb{R}^n : (\alpha, -1) \in \mathcal{L}\}.$$

In particular, the potentially obstructed set is determined by the real linear span of the actual Fourier support. We have

$$s := \dim \mathcal{S} = n + 1 - r.$$

The projection of $\mathcal{S}$ onto the x -coordinates is injective: a nonzero pure vertical symmetry would make Q independent of u , hence zero by its zero vertical mean. Let $E \subset \mathbb{R}^n$ be the image. It is a rational subspace of dimension s . There is a unique $\beta \in E$ such that

$$(L.6) \quad \mathcal{S} = \{(a, \beta \cdot a) : a \in E\}.$$

Choosing a rational basis of E makes the linear system for β rational, so $\beta \in \mathbb{Q}^n$. Put $d = n - s = r - 1$ and choose a rational $d \times n$ matrix B whose rows are a basis of $E^\perp$. Then

$$(L.7) \quad K = \beta + E^\perp = \{\beta + B^T \rho : \rho \in \mathbb{R}^d\}, \quad G = BB^T > 0.$$

The case $d = 0$ is interpreted separately below.

THEOREM L.3 (Exact Fourier-rank reduction). *After passage to a finite periodic cover, there is a smooth periodic potential $\bar{V}(y, z)$ in $d = r - 1$ base variables, with a constant positive-definite kinetic matrix G , such that*

$$(L.8) \quad V^0(x, u) = \bar{V}(Bx, u - \beta \cdot x).$$

For each rational ρ and $\alpha = \beta + B^T \rho$, the original problem has a closed minimizing foliation of slope α if and only if

$$(L.9) \quad G : D_y^2 v = -\bar{V}_z(y, v)$$

has a closed minimizing foliation of slope ρ for the action with density

$$\frac{1}{2} \nabla v^T G \nabla v - \bar{V}(y, v).$$

The correspondence is

$$(L.10) \quad U(x) = \beta \cdot x + v(Bx).$$

Every rational slope outside K is automatic. In particular, the original all-rational condition is equivalent to the all-rational condition for this d -dimensional quotient, allowing its constant kinetic form and its specified period lattice.

Proof. The function $\tilde{V}(x, z) = V^0(x, z + \beta \cdot x)$ is invariant under translation of x by every vector in E , by (L.6). It therefore factors through $y = Bx$, proving (L.8) as an identity on Euclidean spaces. A rational right inverse is $C_B = B^T(BB^T)^{-1}$, and one may explicitly set $\bar{V}(y, z) = V^0(C_B y, z)$, since $\beta \cdot C_B y = 0$.

Choose an integer N for which $N\beta \in \mathbb{Z}^n$ and NB is integral. Then $\widetilde{V}$ is periodic under $x \mapsto x + Nm$ for $m \in \mathbb{Z}^n$, and $\overline{V}$ is periodic under the full-rank lattice

$$\Gamma = B(N\mathbb{Z}^n) \subset \mathbb{R}^d,$$

and under $z \mapsto z + 1$. This is a genuine lattice: B has rational entries and rank d . These assertions follow directly from (L.8); no irrational quotient is being silently identified with a torus. Rational slopes in these coordinates are precisely those made integer-valued on the lattice after a finite cover. They correspond bijectively to $K \cap \mathbb{Q}^n$.

Take an original closed minimizing foliation at $\alpha \in K$. Each leaf is a global cell minimum. Corollary L.2 gives $a \cdot \nabla U = \beta \cdot a$ for every $a \in E$, so $U - \beta \cdot x$ factors through Bx and has the form (L.10). Its equation is (L.9) because

$$\nabla_x U = \beta + B^T \nabla_y v, \quad \Delta_x U = (BB^T) : D_y^2 v.$$

It has the required reduced affine increments and positive transverse derivative. Its bounded deviation and mean height pass to the quotient. The reduced solution foliation calibrates its leaves: in the constant- G version of (A.3), replace p in the linear term by Gp , and replace the square by $(\nabla v - p)^T G (\nabla v - p)/2$. Thus it is a closed minimizing foliation.

Conversely, lift a reduced closed solution foliation using (L.10). It is a solution foliation of the original equation. Its increments close on a finite cover chosen so that the reduced increments and $\beta \cdot x$ increments are integral. Lemma A.4 then supplies all-dimensional class-A minimality and descent to natural rational holonomy on the original torus.

There is also an exact cell variational comparison, independent of that last calibration. Since $B\beta = 0$, the lifted energy is

$$(L.11) \quad E_{V^0}[U] = \frac{1}{2}|\beta|^2 + E_{\overline{V},G}[v].$$

Normalized Haar measure on a compatible original torus pushes forward to normalized Haar measure on the quotient torus, proving this equality without a missing volume factor. Every original global cell minimum inherits the symmetry, hence is a lift. Thus minimizing over all original periodic competitors gives exactly the same infimum as minimizing over the reduced competitors and adding $|\beta|^2/2$. In particular, no restriction of the allowed all-dimensional competitors has been introduced. The reduced force has full Fourier rank $d + 1$: the injective frequency map $(\kappa, m) \mapsto (B^T \kappa - m\beta, m)$ lifts its support to the original support, whose span has dimension $r = d + 1$. The original u -independent part in (L.1) merely subtracts its constant cell average. Finally apply Corollary L.2 outside K . $\square$

COROLLARY L.4 (Rank one is globally excluded). *If $Q \neq 0$ has Fourier rank one, a closed minimizing foliation fails at its single resonant rational slope. Thus rank-one forces cannot satisfy $\text{RH}_{\mathbb{Q}}$, at any amplitude.*

Proof. Here $E = \mathbb{R}^n$, $K = \{\beta\}$ and the reduced base has dimension zero. Every global minimizer at β satisfies $\nabla U = \beta$, so its leaves would have to be affine constants in the reduced variable z . Their equation requires $\bar{V}_z(z) = 0$. A foliation fills every z , forcing $\bar{V}_z = 0$, contrary to $Q \neq 0$. $\square$

Remark L.5 (What happens at rank two and full rank). At rank two the theorem reduces the problem exactly to a one-dimensional periodic Newton variational problem, after rescaling its constant positive kinetic coefficient and its period. No separate one-dimensional rigidity theorem is invoked or reproduced here. At full rank $r = n + 1$, the symmetry space is zero and the reduction leaves the original base dimension unchanged. It does not resolve that case.

Example L.6. For $n \geq 2$, a potential $V(x, u) = F(x_1, u)$ has unused directions $e_2, \dots, e_n$. Every rational slope with $(\alpha_2, \dots, \alpha_n) \neq 0$ automatically has a closed minimizing foliation, for an arbitrary smooth F . The remaining line $\alpha_2 = \dots = \alpha_n = 0$ is exactly the one-dimensional problem for F . Density, and even full measure of a slope set, is therefore not a replacement for all rational tests. Here the exceptional line is exactly the symmetry constraint in Theorem L.1.

L.3. *A quantitative, arbitrary-amplitude obstruction.* The quotient theorem is exact for symmetric potentials. The cell gap also gives a robust result for nearby nonsymmetric potentials.

THEOREM L.7 (Explicit neighborhood of a travelling-wave obstruction). *Let $F : \mathbb{T} \rightarrow \mathbb{R}$ be smooth and nonconstant, and let*

$$V_0(x, u) = A(x) + F(\ell u + k \cdot x), \quad \ell \in \mathbb{N}, \quad k \in \mathbb{Z}^n, \quad \beta = -k/\ell.$$

Use the cubic cell of side ℓ , and let $C_P = \ell/(2\pi)$ be its normalized L^2 Poincare constant. Choose z_* with

$$g_* := \max F - F(z_*) > 0, \quad D_F := \|F'\|_{\infty} > 0,$$

and define

$$(L.12) \quad \delta_* = \min \left\{ \frac{g_*}{2}, \frac{g_*^2}{8\ell^2 C_P^2 D_F^2} \right\} > 0.$$

Then $B_{\beta}(V_0) \geq \delta_*$. Every smooth periodic potential V with

$$(L.13) \quad \|V - V_0\|_{\infty} < \frac{\delta_*}{2}$$

fails the closed rational foliation test at β . The perturbation may have arbitrary Fourier support and need not preserve any symmetry.

Proof. Write a competitor as $U = \beta \cdot x + w(x)$ and put $z = \ell w = \ell U + k \cdot x$. The cross kinetic term has zero average. Relative to the global minimum attained by a constant z maximizing F , the excess energy is exactly

$$(L.14) \quad \frac{1}{2\ell^2} \langle |\nabla z|^2 \rangle_C + \langle \max F - F(z) \rangle_C.$$

Consider the mean constraint $\langle z \rangle_C = z_*$. Set $t = \langle |\nabla z|^2 \rangle_C^{1/2}$. Poincaré and the Lipschitz bound give

$$\langle \max F - F(z) \rangle_C \geq g_* - D_F \langle |z - z_*| \rangle_C \geq g_* - D_F C_P t.$$

If $t \leq g_*/(2D_F C_P)$, the potential part is at least $g_*/2$. Otherwise the nonnegative kinetic part in (L.14) exceeds $g_*^2/(8\ell^2 C_P^2 D_F^2)$. Every competitor of that mean therefore has excess at least δ_* . Some other mean attains excess zero, proving the lower bound on the gap. Now (A.6) gives

$$B_\beta(V) \geq \delta_* - 2\|V - V_0\|_\infty > 0.$$

Apply Theorem A.5. □

For example, with $\ell = 1$ and $F(z) = a \cos(2\pi z)$, $a > 0$, choose $z_* = 1/2$. The explicit bound is

$$\delta_* = \min\{a, 1/2\}.$$

Thus arbitrary smooth perturbations of uniform norm less than $\min\{a/2, 1/4\}$ cannot restore the resonant closed foliation. This is a lower bound on an obstruction, not a claim that the constants or the neighborhood are optimal.

Appendix M. Transition actions and exact source-free identities

This appendix retains the exact reductions from the earlier research notes. No identity is strengthened to an unproved vanishing statement. Unless a rational cover is specified, all physical integrals are over $M = \mathbb{T}^{n+1}$ with normalized Haar measure.

M.1. The calibration conjugacy.

PROPOSITION M.1 (Zero-foliation transition action). *Suppose a smooth zero-slope solution foliation has rigid holonomy. Write it as $U_0(x, t)$, periodic in x , with $U_0(x, t+1) = U_0(x, t) + 1$, and put $\phi = \partial_t U_0 > 0$. Under the invertible graph change*

$$u(x) = U_0(x, s(x)), \quad \sigma(x, t) = \phi(x, t)^2,$$

the Euclidean action is, modulo its closed Hilbert calibration, exactly

$$(M.1) \quad E_\sigma[s] = \frac{1}{2} \int \sigma(x, s) |\nabla s|^2 dx, \quad \int_0^1 \sqrt{\sigma(x, t)} dt = 1.$$

Fixed-trace minimality, bounded-deviation slopes, and rigid deck laws of other foliations are preserved.

Proof. Let $b(x, u)$ be the slope field of U_0 . The Hilbert identity gives

$$\left(\frac{1}{2}|\nabla u|^2 - V(x, u)\right) dx = u^* \omega_b + \frac{1}{2}|\nabla u - b(x, u)|^2 dx.$$

The chain rule gives $\nabla u = b(x, u) + \phi(x, s)\nabla s$, proving the action formula. The Hilbert term is fixed for fixed-trace comparisons. Both the coordinate map and its inverse have bounded derivatives on the periodic quotient, so the Sobolev comparison classes correspond. The identity $\int \phi dt = 1$ is vertical monodromy. Since $U_0(x, t) - t$ is bounded and periodic, slopes are preserved. Composing another foliation with the inverse zero phase directly preserves its deck law. $\square$

The transformed equation is

$$\operatorname{div}(\sigma(x, s)\nabla s) - \frac{1}{2}\sigma_t(x, s)|\nabla s|^2 = 0.$$

For its slope field p on the physical (x, t) torus, this is

$$(M.2) \quad \operatorname{div}_x(\sigma p) + \partial_t \left(\frac{1}{2}\sigma|p|^2 \right) = 0.$$

For an inverse phase T , the corresponding perspective functional is

$$\mathcal{J}_\sigma[T] = \frac{1}{2} \int_M \sigma(x, t) \frac{|\nabla_x T|^2}{T_t} dx dt, \quad T_t > 0.$$

Its first variation gives exactly (M.2); it has a leafwise reparametrization kernel, not an automatically invertible full elliptic Hessian.

M.2. *A raw resonant-square law.*

PROPOSITION M.2. *Let p_α be any solution-foliation field for (M.2). For $k \in \mathbb{Z}^n$, $\ell \in \mathbb{Z} \setminus \{0\}$, and $\beta = -k/\ell$,*

$$(M.3) \quad \sigma \widehat{|p_\alpha - \beta|^2}(k, \ell) = |\beta|^2 \widehat{\sigma}(k, \ell).$$

In particular the left side is independent of the comparison slope whenever several actual foliations exist.

Proof. Test (M.2) by $e^{-2\pi i(k \cdot x + \ell t)}$ and integrate. The result is

$$\int_M \sigma \left(k \cdot p_\alpha + \frac{\ell}{2}|p_\alpha|^2 \right) e^{-2\pi i(k \cdot x + \ell t)} dx dt = 0.$$

Use $k = -\ell\beta$ and complete the square. No differentiation in slope or change of deck character is involved. $\square$

At $\alpha = \beta$ the right side is the Fourier coefficient of a nonnegative deformation-energy density. Its mean is determined by the exact perspective Bregman gap:

$$\mathcal{J}_\sigma[t - \beta \cdot x] - \mathcal{J}_\sigma[T_\beta] = \frac{1}{2} \int_M \sigma |p_\beta - \beta|^2.$$

If $\|\sigma - 1\|_\infty \leq \epsilon < 1$, cohomology gives $\int q = 1$, $\int qp_\beta = \beta$, and Cauchy-Schwarz gives $\int q |p_\beta|^2 \geq |\beta|^2$. Comparing the actual and affine phase actions therefore yields

$$(M.4) \quad \int_M \sigma |p_\beta - \beta|^2 \leq 2\epsilon |\beta|^2.$$

This estimate is denominator-uniform, but is only linear in ϵ for the squared defect. It is not a nonlinear contraction for the Fourier coefficient in (M.3).

For comparison, in the original Euclidean variables every rigid phase of the resonant slope satisfies the exact identity

$$(M.5) \quad \widehat{Q}(k, \ell) = \pi i \ell \int_M |p_\beta - \beta|^2 e^{-2\pi i(k \cdot x + \ell u)} dm.$$

It follows by testing $\operatorname{div}_x p_\beta + \partial_u(|p_\beta|^2/2) = Q$ with the same physical character. The constant term in the completed square integrates to zero since $\ell \neq 0$.

M.3. Pairwise and mixed phase identities, with their correct scope. Let τ_r be finitely many admissible phases of actual foliations for one force Q , and set $q_r = (\tau_r)_u$, $p_r = -\nabla_x \tau_r / q_r$. We require every test used below to descend to the integration torus. Arbitrary smooth periodic functions of the phases are permitted for integer slopes, or on a common rational cover for rational slopes. They are not automatically permitted for arbitrary irrational slopes.

PROPOSITION M.3 (Pairwise phase-flatness). *For two actual fields and every legal one-phase test g ,*

$$\int g'(\tau_b) q_b |p_a - p_b|^2 = 0.$$

More generally, for a legal $H(\tau_a, \tau_b)$,

$$\int [q_a H_1(\tau_a, \tau_b) - q_b H_2(\tau_a, \tau_b)] |p_a - p_b|^2 = 0.$$

Proof. Subtract the two residual equations. Their difference is the divergence of $(p_a - p_b, (|p_a|^2 - |p_b|^2)/2)$. Test it by H , use $d\tau_r = q_r(du - p_r \cdot dx)$, and integrate. The two coefficients after completing squares are $-|p_a - p_b|^2/2$ and $|p_a - p_b|^2/2$. The first assertion follows by taking H to depend only on its second argument. $\square$

PROPOSITION M.4 (Mixed chain rule, not an annihilation theorem). *For a legal $F \in C^\infty(\mathbb{T}^N)$,*

$$(M.6) \quad \int_M Q_u F(\tau) dm = \frac{1}{2} \sum_{r,s} \int_M F_{rs}(\tau) q_r q_s |p_r - p_s|^2 dm.$$

Proof. The continuity equation gives $\nabla_x q_r = -p_r(q_r)_u - q_r(p_r)_u$. Substitution into a product derivative yields

$$q_r Q = \operatorname{div}_x(q_r p_r) + \partial_u(q_r |p_r|^2).$$

Multiply by $F_r(\tau)$ and sum. After integration by parts, using $\nabla_x F_r = -\sum_s F_{rs} q_s p_s$ and $\partial_u F_r = \sum_s F_{rs} q_s$, we obtain

$$\int Q \partial_u F(\tau) = \sum_{r,s} \int F_{rs} q_r q_s (p_r \cdot p_s - |p_r|^2).$$

Symmetrization gives minus one half the squared distance. Integrating the left side by parts in u proves (M.6). $\square$

There is also a pointwise differential-form version. Put $\Omega = dx_1 \wedge \cdots \wedge dx_n \wedge du$, $Y_r = (q_r p_r, q_r |p_r|^2)$, $\eta_r = \iota_{Y_r} \Omega$, and $\lambda_0 = \iota_{\partial_u} \Omega$. Then

$$\left[\frac{1}{2} \sum_{r,s} F_{rs} q_r q_s |p_r - p_s|^2 - F Q_u \right] \Omega = d \left[\sum_r F_r \eta_r - Q F \lambda_0 \right].$$

This follows by the same product rule, using $d\eta_r = q_r Q \Omega$ and $d\tau_s(Y_r) = q_r q_s (|p_r|^2 - p_r \cdot p_s)$. Stokes' theorem gives (M.6), not vanishing of either side. If the phase-cylinder algebra is point-separating and dense, vanishing for all such F would already imply $Q_u = 0$. It is not an additional polarization step supplied by Proposition M.3.

M.4. Leafwise Hessian and Haar factorizations.

PROPOSITION M.5 (Exact periodic trace-free Hessian identity). *For a rational closed leaf U , put $A = D^2 U$ and $S = A - (\Delta U)I/n$. Then*

$$\int_C |D^2 U|^2 = \int_C (\Delta U)^2, \quad \int_C |S|^2 = \frac{n-1}{n} \int_C Q(x, U)^2.$$

Proof. The affine part of U has zero Hessian. For its periodic corrector, integration by parts twice gives $\int U_{ij}^2 = \int U_{ii} U_{jj}$. Sum and use the orthogonal trace decomposition. This is the standard periodic Hessian identity, not a new vanishing mechanism. $\square$

For a fixed phase write $D_i = \partial_{x_i} + p_i \partial_u$ and $r = p_u$. Differentiating the residual gives

$$Q_u = \sum_i D_i r_i + |r|^2, \quad \operatorname{div}_{x,u} D_i = r_i.$$

Hence for every smooth periodic η ,

$$(M.7) \quad \int_M \left[\sum_i |D_i \eta|^2 + \frac{1}{2} Q_u \eta^2 + \frac{1}{4} |r|^2 \eta^2 \right] = \int_M \sum_i \left| D_i \eta - \frac{1}{2} r_i \eta \right|^2.$$

Indeed the cross term on the right is $-\frac{1}{2} \int r_i D_i(\eta^2) = \frac{1}{2} \int \eta^2 (D_i r_i + r_i^2)$. The formula is a nonnegative factorization, but its two potential terms cannot be separated into independent signs.

M.5. *High integer slopes and separation of phase characters.* The next statement retains only the norms justified directly by the fixed-cell equations. In particular it does not claim high physical-coordinate regularity uniformly in the slope.

PROPOSITION M.6 (Uniform integer-slope flattening). *Suppose smooth rigid solution foliations exist at a sequence of integer slopes $a \in \mathbb{Z}^n$ with $|a| \rightarrow \infty$. Normalize their leaf parameter by the spatial mean and write*

$$U_a(x, t) = a \cdot x + t + w_a(x, t), \quad \int_{\mathbb{T}^n} w_a(x, t) dx = 0.$$

For every $0 < \gamma < 1$,

$$(M.8) \quad \sup_t \|w_a(\cdot, t)\|_{C^{1,\gamma}(\mathbb{T}^n)} \longrightarrow 0, \quad \sup_t \|\partial_t U_a(\cdot, t) - 1\|_{C^{1,\gamma}(\mathbb{T}^n)} \longrightarrow 0.$$

For the corresponding mean-normalized inverse phases,

$$\|\tau_a - (u - a \cdot x)\|_{C^0(M)} \longrightarrow 0, \quad \|q_a - 1\|_{C^0(M)} \longrightarrow 0.$$

Proof. The mean parametrization is an increasing degree-one diffeomorphism, since $U_t > 0$. Integer slope makes w_a periodic on the fixed unit cell. Its equation and normalization are

$$\Delta w_a = Q(x, a \cdot x + t + w_a), \quad \int w_a = 0.$$

The right side is uniformly bounded. The zero-mean periodic Poisson estimate therefore bounds w_a in every fixed $W^{2,p}$, uniformly in a, t . Choosing p large makes these families precompact in the displayed $C^{1,\gamma}$ space.

We justify the oscillatory limit, including its uniformity. Expand the smooth zero-vertical-mean force in its absolutely convergent Fourier series, and first retain finitely many modes (k, m) , all with $m \neq 0$. For a smooth periodic test ζ , each retained integral has the form

$$e^{2\pi i m t} \int_{\mathbb{T}^n} \zeta(x) e^{2\pi i (k+ma) \cdot x} e^{2\pi i m w_a(x, t)} dx.$$

Integration by parts in the constant direction $(k+ma)/|k+ma|^2$ bounds it by $C_{k,m,\zeta}/|k+ma|$, using the uniform C^1 bound for w_a . The Fourier tail is uniformly small. Thus the right side of the Poisson equation tends to zero

distributionally, uniformly in t . Every subsequential limit of $w_a(\cdot, t_a)$ is harmonic with mean zero, and hence vanishes. Compactness proves the first limit in (M.8).

Put $\phi_a = \partial_t U_a > 0$. It satisfies

$$\Delta \phi_a = Q_u(x, U_a) \phi_a, \quad \int_{\mathbb{T}^n} \phi_a = 1.$$

The fixed-cell Harnack inequality for this bounded zero-order coefficient gives uniform positive upper and lower bounds. Periodic elliptic estimates then bound ϕ_a in every fixed $W^{2,p}$. The same Fourier argument applies to $Q_u(x, U_a) \phi_a$: in each retained mode the amplitude is now $\zeta \phi_a e^{2\pi i m w_a}$, still uniformly C^1 . Since Q_u also has zero vertical mean, every limit of ϕ_a is harmonic with mean one. It equals one. This proves the second limit.

At fixed (x, u) , inversion gives $\tau_a = u - a \cdot x - w_a(x, \tau_a)$ and $q_a = 1/\phi_a(x, \tau_a)$. The uniform C^0 conclusions follow. Differentiating these identities in physical x introduces the large slope; no stronger physical-coordinate conclusion is made. $\square$

COROLLARY M.7 (A separating phase algebra, not a vanishing theorem). *If rigid solution foliations exist at every integer slope, their phase characters generate a uniformly dense self-adjoint algebra in $C(M)$. Under this hypothesis the assertion that the right side of (M.6) vanishes for every finite integer-slope family and every periodic smooth F is equivalent to $Q_u = 0$, and hence to $Q = 0$.*

Proof. Use the mean-normalized phases. Suppose two points have equal characters $e^{2\pi i \tau_a}$ for every integer a . Taking the quotient of the characters for $a = Ne_j$ and $a = (N+1)e_j$ and letting $N \rightarrow \infty$ recovers equality of $e^{2\pi i x_j}$ at the two points, for every j . With x fixed on the torus, any one strictly increasing degree-one phase then gives equality of u modulo one. Thus the characters separate points. Stone–Weierstrass applies to their unital self-adjoint algebra. Relabelling a phase by a degree-one circle diffeomorphism does not alter joint point separation.

By (M.6), the asserted vanishing means that $Q_u dm$ annihilates every smooth phase cylinder, hence every monomial in that algebra. Density implies $Q_u = 0$. Conversely $Q_u = 0$ makes the mixed identity vanish. Finally the vertical mean of Q is zero, so vertical constancy means $Q = 0$. This equivalence is why the mixed vanishing cannot be inserted as an already established polarization lemma. $\square$

M.6. Weighted Riccati identities and the shear term. For one actual smooth foliation put $D_i = \partial_{x_i} + p_i \partial_u$, $r = p_u$, and $A_{ij} = D_i p_j$. Then A is symmetric,

$\text{tr } A = -V_u$, and

$$Q_u = \sum_i D_i r_i + |r|^2, \quad \text{div}_{x,u} D_i = r_i.$$

For any $\lambda \neq 0$, multiplication by $e^{-\lambda V}$ and physical integration by parts give

$$(M.9) \quad \int_M e^{-\lambda V} [V_u^2 + r \cdot (\nabla_x V + p V_u)] dm = 0.$$

Indeed the $|r|^2$ term cancels the divergence of D_i , leaving $\int e^{-\lambda V} Q_u = \lambda \int e^{-\lambda V} r \cdot DV$; vertical integration of $Q_u = -V_{uu}$ gives the negative of $\lambda \int e^{-\lambda V} V_u^2$.

A more general scalar weight records the dimensional shear explicitly. Put $W_\eta = \exp(-|p|^2/2 - \eta V)$. The same integration gives

$$(M.10) \quad \int_M W_\eta [r \cdot Ap + \eta r \cdot \nabla_x V + (\eta + 1)V_u p \cdot r + \eta V_u^2] dm = 0.$$

To check this, the Riccati equation gives $\int W_\eta Q_u = \int W_\eta r \cdot D(|p|^2/2 + \eta V)$, whereas vertical integration gives $\int W_\eta Q_u = -\int W_\eta V_u (p \cdot r + \eta V_u)$. Set $c_n = (n-1)/n$ and $S = A + (V_u/n)I$. Substituting $\eta = -c_n$ in (M.10) cancels the trace terms and yields

$$(M.11) \quad c_n \int_M e^{-|p|^2/2 + c_n V} V_u^2 dm = \int_M e^{-|p|^2/2 + c_n V} r \cdot (Sp - c_n \nabla_x V) dm.$$

No sign is asserted for the right side. The scalar shear transgression in Appendix N explains why a signed slope-space integration does not supply an additional positive remainder.

Appendix N. Slope-chart moments, degree, and a sharp equality criterion

This appendix concerns an auxiliary setting with an explicit additional regularity hypothesis. Let $O \subset \mathbb{R}^n$ be open and suppose

$$\tau_\alpha = u - \alpha \cdot x + \psi_\alpha(x, u), \quad (\alpha, x, u) \mapsto \psi_\alpha(x, u) \text{ is smooth on } O \times M,$$

is a family of admissible solution phases for the same V . Their densities need not agree. The smooth slope chart is assumed here, not deduced from separate existence of foliations. No result in the main common-volume proof uses this appendix.

N.1. *Differentiating the deck law.* Before using a slope derivative, we record the deck law it satisfies. This distinguishes a smooth chart of rigid foliations from a stronger affine ansatz for the leaves themselves.

LEMMA N.1 (The Jacobi factor in a slope derivative). *Let $a \mapsto U_a$ be a C^1 family in the slope parameter, smooth in (x, t) , satisfying*

$$U_a(x + m, t) = U_a(x, t + a \cdot m), \quad m \in \mathbb{Z}^n.$$

Assume that the mixed derivatives used below exist. Put $W_j = \partial_{a_j} U_a|_{a=a_0}$ and $\varphi = \partial_t U_{a_0}$. Then

$$(N.1) \quad W_j(x + m, t) = W_j(x, t + a_0 \cdot m) + m_j \varphi(x, t + a_0 \cdot m).$$

In particular, at $a_0 = 0$ the increment is $m_j \varphi(x, t)$, not m_j . If every leaf solves $\Delta U_a = Q(x, U_a)$, then, at this zero-slope foliation,

$$(N.2) \quad \operatorname{div}_x \left(\varphi^2 \nabla_x \frac{W_j}{\varphi} \right) = 0, \quad \frac{W_j}{\varphi}(x + m, t) = \frac{W_j}{\varphi}(x, t) + m_j.$$

Proof. Differentiate the rigid deck identity at fixed (x, m, t) . The derivative of the shifted phase $t + a \cdot m$ gives the second term in (N.1). At slope zero, both W_j and φ solve

$$\Delta_x w = Q_u(x, U_0)w.$$

Hence

$$\operatorname{div}_x \left(\varphi^2 \nabla_x \frac{W_j}{\varphi} \right) = \varphi \Delta_x W_j - W_j \Delta_x \varphi = 0.$$

The function φ is positive and periodic in x . Dividing (N.1) by it gives the remaining assertion. $\square$

Example N.2 (A smooth local slope chart with nonzero force). Take $\beta = e_1$ in Theorem 3.3. The resulting potential is

$$V(x, u) = -\frac{1}{2} \sin^4(\pi(u - x_1)).$$

Its nonexceptional rotating profiles form a smooth slope chart on a neighbourhood of $a = 0$, while $V_u \neq 0$. Indeed, the transformed slope is $a - e_1$, bounded away from zero there. The period map has strictly negative derivative, so the selected energy depends smoothly on $|a - e_1|$. The normalized primitive and its inverse are then smooth in (a, x, t) by the implicit-function theorem. The corresponding inverse-phase corrections are periodic in (x, u) . Thus local smoothness in slope, with the actual rigid deck law, does not by itself force rigidity. An ansatz giving $W_j(x + m, t) = W_j(x, t) + m_j$ would impose an additional restriction; it is not a consequence of rigid holonomy.

N.2. *Matrix moments and degree.* Define

$$\mathcal{A}(\alpha) = \int_M q_\alpha \left(\frac{1}{2} |p_\alpha|^2 - V \right) dm, \quad B_\alpha = D_\alpha p_\alpha, \quad H_\alpha = D_\alpha^2 \mathcal{A}(\alpha).$$

Here $(B_\alpha)_{ij} = \partial_{\alpha_j} p_{\alpha, i}$.

LEMMA N.3 (Two matrix moments). *On this chart,*

$$(N.3) \quad \nabla_\alpha \mathcal{A} = \int_M p_\alpha dm, \quad H_\alpha = \int_M B_\alpha dm = \int_M q_\alpha B_\alpha^T B_\alpha dm,$$

and

$$(N.4) \quad \int_M q_\alpha^2 B_\alpha dm = I_n.$$

Proof. A slope variation by h changes the phase form by $-h \cdot dx + d(\partial_h \psi_\alpha)$. The periodic exact part pairs to zero with the divergence-free first derivative of the phase integrand. This gives

$$D\mathcal{A}(\alpha)[h] = h \cdot \int_M p_\alpha dm,$$

and differentiation gives $H_\alpha = \int B_\alpha$. For two nearby slopes, the perspective Bregman identity gives the exact relation

$$(N.5) \quad \mathcal{A}(\beta) - \mathcal{A}(\alpha) - (\beta - \alpha) \cdot \int_M p_\alpha dm = \frac{1}{2} \int_M q_\beta |p_\beta - p_\alpha|^2 dm.$$

Indeed, the difference of the two phase forms has constant cohomological part $(-\beta + \alpha, 0)$ and periodic exact part, which gives precisely the displayed linear term. Set $\beta = \alpha + th$, divide by $t^2/2$, and let $t \rightarrow 0$. Smoothness proves $h^T H_\alpha h = \int q_\alpha |B_\alpha h|^2$, hence the last equality in (N.3).

For the second moment put $Y_j = -\partial_{\alpha_j} \tau_\alpha = x_j - \partial_{\alpha_j} \psi_\alpha$. Differentiating $D_i^\alpha \tau_\alpha = 0$ gives

$$(N.6) \quad D_i^\alpha Y_j = q_\alpha (B_\alpha)_{ij}.$$

The vector field $q_\alpha D_i^\alpha$ is divergence-free on M . Since $Y_j - x_j$ is periodic,

$$\int_M q_\alpha^2 (B_\alpha)_{ij} dm = \int_M q_\alpha D_i^\alpha Y_j dm = \delta_{ij} \int_M q_\alpha dm = \delta_{ij}.$$

This proves (N.4). $\square$

LEMMA N.4 (Cohomological degree identity). *At every slope of the chart,*

$$(N.7) \quad \int_M q_\alpha^{n+1} \det B_\alpha dm = 1.$$

No sign condition on $\det B_\alpha$ is required.

Proof. Suppress α . Differentiation of $\nabla_x \tau = -q p$ yields

$$dY_j = q \sum_i B_{ij} dx_i - \frac{\partial_{\alpha_j} q}{q} d\tau.$$

Wedge these n identities with $d\tau$. Terms with two factors $d\tau$ vanish, giving

$$dY_1 \wedge \cdots \wedge dY_n \wedge d\tau = q^{n+1} \det B dx_1 \wedge \cdots \wedge dx_n \wedge du.$$

The forms on the left are periodic and closed. Their cohomology classes are $[dY_j] = [dx_j]$ and $[d\tau] = [du - \alpha \cdot dx]$, whose cup product integrates to one. This proves (N.7). The argument uses closed forms, not a circle-valued phase at an arbitrary real slope. $\square$

THEOREM N.5 (Slope-entropy square and its equality case). *Put $M_3(\alpha) = \int_M q_\alpha^3 dm$. Then*

$$(N.8) \quad H_\alpha - \frac{1}{M_3(\alpha)} I = \int_M q_\alpha \left(B_\alpha - \frac{q_\alpha}{M_3(\alpha)} I \right)^T \left(B_\alpha - \frac{q_\alpha}{M_3(\alpha)} I \right) dm.$$

Consequently $H_\alpha \succeq M_3(\alpha)^{-1} I$. For $n \geq 2$, equality in

$$(N.9) \quad \det H_\alpha \geq M_3(\alpha)^{-n}$$

at one slope forces $V_u = 0$. The bound is attained by affine phases of a u -independent potential.

Proof. Expand the right side of (N.8). Equations (N.3) and (N.4) give the three terms

$$H_\alpha - \frac{2}{M_3} I + \frac{1}{M_3} I,$$

proving the identity. If the determinant inequality is equality, all eigenvalues of H_α must equal M_3^{-1} . Positivity of q_α then makes the square vanish pointwise, so

$$B_\alpha = \frac{q_\alpha}{M_3} I.$$

The degree identity becomes

$$(N.10) \quad \int_M q_\alpha^{2n+1} dm = \left(\int_M q_\alpha^3 dm \right)^n.$$

Regard $q_\alpha dm$ as a probability measure. Jensen's inequality for the strictly convex function $t \mapsto t^n$, applied to $t = q_\alpha^2$ and $n \geq 2$, gives the inequality corresponding to (N.10). Equality forces q_α^2 to be constant. Thus $q_\alpha = 1$, and Lemma 5.3 gives $V_u = 0$. Affine phases have $q = 1$, $B = I$ and $H = I$, proving attainability. $\square$

This is an equality criterion, not a proof that equality is attained by an arbitrary complete integral. The chart hypothesis and the equality hypothesis cannot be removed by citing the moment identities themselves. In particular, the signed degree formula alone provides neither a positive slope-space measure nor a nonfolding theorem.

N.3. *Weighted Piola identities and the fold term.* The following formulas require a C^2 local slope chart $a \mapsto p_a(x, u)$; this hypothesis is not derived from separate rational foliations. Set

$$\begin{aligned} D_i &= \partial_{x_i} + p_i \partial_u, & r &= p_u, & A_{ij} &= D_i p_j, \\ B &= D_a p, & J &= \det B, & C &= \text{cof } B. \end{aligned}$$

The matrix A is symmetric by graph integrability. All identities in this section are local in slope, and no global change of variables in a is assumed.

PROPOSITION N.6 (Full weighted Piola balance). *For every smooth $g : \mathbb{R}^n \rightarrow \mathbb{R}$,*

$$(N.11) \quad \partial_u(g(p)J) = \text{div}_a(g(p)C^T r),$$

$$(N.12) \quad \partial_{x_i}(g(p)J) + \partial_u(p_i g(p)J) = \text{div}_a(g(p)C^T A_i),$$

where $A_i = (A_{i1}, \dots, A_{in})^T$.

Proof. The Piola identity is $\partial_{a_s} C_{js} = 0$. Differentiating $A_{ij} = D_i p_j$ gives

$$D_i B_{js} = \partial_{a_s} A_{ij} - B_{is} r_j.$$

Contract with C_{js} , use $CB^T = JI$, and obtain $D_i J + J r_i = \text{div}_a(C^T A_i)$. This is (N.12) for $g = 1$. Likewise $\partial_u B_{js} = \partial_{a_s} r_j$ gives (N.11) for $g = 1$. For the weighted formulas, $D_a[g(p)] = B^T \nabla_p g$ and $(B^T \nabla_p g) \cdot C^T X = J \nabla_p g \cdot X$. Product differentiation now proves both displays. The calculations are polynomial in B, C , so no division by J and no exclusion of its zero set occurred. $\square$

If, in addition, the map $a \mapsto p_a(x, u)$ is proper of degree one on all $\mathbb{R}^n$, the signed area formula gives

$$\int_{\mathbb{R}^n} g(p_a(x, u)) J_a(x, u) da = \int_{\mathbb{R}^n} g(p) dp \quad (g \in C_c^\infty(\mathbb{R}^n)).$$

The properness and degree are assumptions in this statement, not conclusions of Proposition N.6. This is a signed identity; it does not supply a positive invariant measure.

At a regular fold set $\Sigma_{x,u} = \{a : J_a(x, u) = 0\}$ with $\nabla_a J \neq 0$, let ν_a point from $J < 0$ to $J > 0$. Distributionally in the slope variable,

$$\nabla_a \text{sgn } J = 2\nu_a \mathcal{H}^{n-1} \llcorner \Sigma_{x,u}.$$

The chain rule for $|J|$ and (N.11) therefore give

$$(N.13) \quad \partial_u(g(p)|J|) = \text{div}_a(\text{sgn } J g(p)C^T r) - 2g(p)(C^T r \cdot \nu_a) \mathcal{H}^{n-1} \llcorner \Sigma_{x,u}.$$

The analogous horizontal formula replaces r by A_i on the slope side and has left side $\partial_{x_i}(g|J|) + \partial_u(p_i g|J|)$. These identities follow by multiplying the signed formula by $\text{sgn } J$; derivatives of the sign in physical variables multiply J and vanish. The extra slope flux in (N.13) has no sign supplied by the hypotheses.

N.4. *A scalar shear transgression and its limitation.* On a regular chart, put $S = A - QI/n$ and $D_{ij} = \partial_{p_j} r_i$. The division-free interpretation is

$$JD_{ij} = C_{js} \partial_{a_s} r_i.$$

For any smooth weight $\Phi(x, u, p)$ for which the outer slope-boundary terms vanish, integration by parts gives

$$(N.14) \quad \int \Phi JS : D = \int J \left[\Phi(Q_u - |r|^2) + r \cdot \left(\nabla_x \Phi + p \Phi_u + \frac{Q}{n} \nabla_p \Phi \right) \right].$$

Here and in this paragraph integration is over the physical torus and the slope domain. To verify the identity, begin with $\int \Phi S_{ij} C_{js} \partial_{a_s} r_i$ and integrate in a . Use

$$C_{js} \partial_{a_s} S_{ij} = D_i J + J r_i, \quad \sum_i D_i r_i = Q_u - |r|^2,$$

then integrate $D_i J$ in physical variables, whose divergence contributes r_i . The $A \nabla_p \Phi$ and $S \nabla_p \Phi$ terms leave $(Q/n) \nabla_p \Phi$, proving (N.14).

For the Gaussian weight $W = \exp(-|p|^2/2 - V/n)$, a noncompact slope integral requires an additional justification. We give the cutoff identity first.

Suppose the chart is defined on all $\mathbb{R}^n$, the map

$$(x, u, a) \mapsto (x, u, p_a(x, u))$$

is proper, and each fibre map $a \mapsto p_a(x, u)$ has degree one. The properness here is joint in the physical and slope variables. Choose $\rho_R \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \rho_R \leq 1$, equal to one on $|p| \leq R$, supported on $|p| \leq 2R$, and $|\nabla \rho_R| \leq C/R$. Then (N.14), applied to $\Phi = \rho_R(p)W$, gives

$$(N.15) \quad \begin{aligned} & \int \rho_R(p) W J \left(\frac{1}{n} V_u^2 + |r|^2 + \frac{1}{n} r \cdot \nabla_x V + S : D \right) \\ &= \frac{1}{n} \int W J Q r \cdot \nabla_p \rho_R(p). \end{aligned}$$

All integrals in this display have compact support in slope. As before, $JS : D$ is interpreted by its division-free cofactor expression at singular points of the chart.

To verify the formula, the derivative terms in (N.14) satisfy

$$p \Phi_u + \frac{Q}{n} \nabla_p \Phi = \frac{Q}{n} W \nabla_p \rho_R, \quad \nabla_x \Phi = -\frac{1}{n} \Phi \nabla_x V,$$

where these are partial derivatives with p fixed. After rearranging, the remaining term is

$$\int \rho_R(p) W J \left(Q_u + \frac{1}{n} V_u^2 \right).$$

The signed degree formula makes this equal to

$$- \left(\int_{\mathbb{R}^n} \rho_R(p) e^{-|p|^2/2} dp \right) \int_M \partial_u (e^{-V/n} V_u) dm = 0.$$

This proves (N.15) without discarding a boundary contribution.

For example, the uncut identity follows if, in addition,

$$(N.16) \quad \int W|J|(1+|r|^2) < \infty, \quad \int W|JS : D| < \infty.$$

The right-hand side of (N.15) then tends to zero by Cauchy–Schwarz and the bound on $\nabla \rho_R$. Dominated convergence on the left gives

$$\int WJ \left(\frac{1}{n} V_u^2 + |r|^2 + \frac{1}{n} r \cdot \nabla_x V + S : D \right) = 0.$$

Properness and degree alone are not being used to infer (N.16). This is a conditional compensation identity, not a positive remainder or a global rigidity argument. Replacing J by $|J|$ also retains the fold contribution in (N.13).

N.5. *Exact affine-cover covariance.* Suppose a family with rigid phases is given, and let $N \in \mathbb{N}$, $m \in \mathbb{Z}^n$. Define

$$\begin{aligned} V^{N,m}(x, v) &= N^{-2} V(x, Nv + m \cdot x), \\ U_\gamma^{N,m}(x, t) &= N^{-1} (U_{N\gamma+m}(x, Nt) - m \cdot x). \end{aligned}$$

Direct substitution proves the equation, ordering, and rigid deck laws. The kinetic cross term from the shear is a divergence, so fixed-trace minimality is preserved after multiplying the action by N^{-2} . The inverse phase is

$$\tau_\gamma^{N,m}(x, v) = N^{-1} \tau_{N\gamma+m}(x, Nv + m \cdot x).$$

Consequently, with the original fields evaluated at $(x, Nv + m \cdot x)$,

$$\begin{aligned} q_\gamma^{N,m} &= q_{N\gamma+m}, & p_\gamma^{N,m} &= (p_{N\gamma+m} - m)/N, \\ B_\gamma^{N,m} &= B_{N\gamma+m}, & r_\gamma^{N,m} &= r_{N\gamma+m}. \end{aligned}$$

The leaf Hessian and its trace-free part scale by N^{-1} . In the pure vertical case $m = 0$, at the covered slope $\gamma = a/N$, Haar preservation by $u \mapsto Nu$ gives, for every finite $s \geq 1$,

$$\left\| (q_{a/N}^{N,0})^{1+1/s} B_{a/N}^{N,0} \right\|_{L^s} = \left\| q_a^{1+1/s} B_a \right\|_{L^s},$$

and the signed degree density becomes $(q_a^{n+1} \det B_a)(x, Nu)$. Its integral is unchanged. Boundedness of these norms and preservation of signed degree do not force strong convergence: a nonconstant vertical profile can persist as pure oscillation. No compactness-to-rigidity step is inferred from these equalities.

Appendix O. Scalar regularity needed by the selector

For clarity we record the standard regularity input, including why the concrete integrand satisfies it. No regularity of an independently chosen slope family is used.

LEMMA O.1. *Let $J \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ satisfy $\lambda I \preceq J_{\xi\xi} \preceq \Lambda I$ and $|J_{x\xi}(x, \xi)| \leq C(1 + |\xi|)$. If $f \in C^\infty(\mathbb{T}^n)$ has zero mean, then the unique mean-zero minimizer of $\int J(x, \nabla g) - fg$ is smooth. Its smooth norms are locally controlled by the data and its H^1 bound.*

Proof. The first variation is $-\operatorname{div}_x J_\xi(x, \nabla g) = f$. Difference quotients tested against their own cutoffs give $g \in H^2$ locally (globally on the torus): the elliptic main term controls second derivatives and the mixed derivative bound controls the error by $C(1 + |\nabla g|)$. Let $v = (1 + |\nabla g|^2)^{1/2}$. Differentiating in x_j in the weak sense yields

$$\begin{aligned} -\operatorname{div}_x(M\nabla g_j) &= f_j + \operatorname{div}_x B_j, \\ M &= J_{\xi\xi}(x, \nabla g), \quad B_j = J_{x_j\xi}(x, \nabla g), \quad |B_j| \leq Cv. \end{aligned}$$

We spell out the gradient estimate. First use truncated versions of $g_j v^{p-2}$ as test functions, sum in j , and then remove the truncation. The principal terms are

$$\sum_j \int v^{p-2} M \nabla g_j \cdot \nabla g_j + (p-2) \int v^{p-2} M \nabla v \cdot \nabla v.$$

They are bounded below by

$$\lambda \int v^{p-2} |D^2 g|^2 + \lambda(p-2) \int v^{p-2} |\nabla v|^2.$$

The divergence terms involving B_j are bounded by $Cp \int v^{p-1} |D^2 g|$, and the terms involving f_j by $C \int v^p$, since $v \geq 1$. Young's inequality absorbs half of the first principal term and leaves $Cp^2 \int v^p$. Since $|\nabla v| \leq |D^2 g|$, we obtain, for $p \geq 2$,

$$(O.1) \quad \int |\nabla(v^{p/2})|^2 \leq Cp^4 \int v^p.$$

The difference-quotient argument justifying the differentiated equations gives the same estimates for the truncated tests, so no prior boundedness of ∇g is used. Sobolev iteration gives

$$\|\nabla g\|_\infty \leq C\|(1 + |\nabla g|^2)^{1/2}\|_2.$$

In dimension two, use any fixed finite Sobolev exponent; in dimension one, use the one-dimensional embedding.

The equations for each g_j are now uniformly elliptic scalar divergence-form equations with bounded divergence data. De Giorgi–Nash estimates give $g_j \in C^{0,\vartheta}$ for some $\vartheta > 0$. Then M and B_j are Hölder continuous, and the divergence-form Schauder estimate gives $g_j \in C^{1,\vartheta}$. Subsequent differentiation and Schauder estimates bootstrap to smoothness. Compactness of the torus removes boundary issues. These are standard scalar elliptic steps; see [6]. The calculation (F.21) verifies their growth hypotheses for the selector. $\square$