

Variational rigidity for alternating symplectic billiards

By ROBERT JENKINSON ALVAREZ

Abstract

The task is to classify smooth positively curved pairs of planar convex curves for which every finite physical orbit of the alternating symplectic billiard has non-positive fixed-endpoint second variation. Such pairs are the independently translated homothetic ellipses, and all of their nontrivial finite Hessians are negative definite. The proof combines an exact finite-block averaging identity with a common area-preserving normalization and independent scaling. The same identity bounds the orbit length at which a positive second variation occurs outside the classified family. An exact polar reduction and a Schur-complement identity transfer the theorem to Minkowski billiards, giving a classification of totally integrable metric-table pairs for strongly convex gauges. In the self-Minkowski case, total integrability forces the gauge to come from an inner product. No symmetry or positive-curvature hypothesis is imposed in this case and curvature is obtained from calibration of continuous invariant graphs.

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1. Introduction

The alternating symplectic billiard associates a variational system to two convex curves in the plane. Successive vertices lie alternately on the two curves, and the chord joining the neighbors of each vertex is parallel to the tangent at that vertex. The action is a sum of oriented symplectic areas. We prove that nonpositive second variation along every finite physical orbit determines both curves: they must be homothetic ellipses, with a common linear shape but independent centers and sizes. The two curves need not coincide or satisfy a polarity relation.

The classification also has a finite-orbit counterpart. An exact averaging identity detects a positive second variation outside the elliptic class and gives an explicit upper bound on the length needed to detect it. The rigidity theorem and this finite obstruction are proved before imposing a metric or a relation between the curves. Minkowski billiards enter through a polar reduction of the alternating action; self-Minkowski rigidity is a consequence of that general classification.

1.1. *The pair theorem.* Let Q and P be positively oriented smooth closed convex curves of positive curvature, and let $\omega(x, y) = \det(x, y)$. A physical orbit is an alternating sequence

$$\dots, q_i, p_i, q_{i+1}, p_{i+1}, \dots, \quad q_i \in Q, \quad p_i \in P,$$

whose two-step chords are positive multiples of the intervening oriented tangents. Equivalently, it is a critical configuration of

$$\sum_i [\omega(q_i, p_i) + \omega(p_i, q_{i+1})]$$

on the phase components where the symplectic products of consecutive tangents are positive. Section 2 defines these components and their global dynamics.

For a finite segment we fix its two endpoint vertices and vary the internal vertices along their respective curves. The Hessian is taken in these internal scalar parameters. Its inertia is independent of parametrization at a critical configuration.

THEOREM 1.1 (Variational rigidity of a convex pair). *Let $Q, P \subset \mathbb{R}^2$ be smooth strictly convex closed curves of positive curvature. The following conditions are equivalent:*

- (i) *Along every finite physical alternating orbit segment, the fixed-endpoint Hessian of the alternating action is negative semidefinite.*
- (ii) *Along every such segment with an internal vertex, that Hessian is negative definite.*

(iii) *There are $A \in \mathrm{SL}(2, \mathbb{R})$, $c_Q, c_P \in \mathbb{R}^2$, and $r_Q, r_P > 0$ such that*

$$AQ = c_Q + r_Q S^1, \quad AP = c_P + r_P S^1.$$

Thus the admissible pairs are exactly the independently translated homothetic ellipses with a common linear shape.

The hypothesis in (i) is purely variational. In particular, the theorem does not assume an invariant foliation, differentiability of invariant curves, a first integral, reversibility, or a common center. The passage from semidefiniteness to definiteness uses only extension of a finite segment and the nonzero nearest-neighbor coefficients of its Hessian. The geometric part of the proof uses semidefiniteness alone.

The special case $Q = P$ recovers the elliptic equality class of the one-curve symplectic problem. For two curves there is an additional freedom: dilating either curve separately multiplies the action by a positive scalar, while leaving its physical orbit configurations unchanged after the natural identification. This freedom is the step that turns the Hopf inequality into a product-isoperimetric equality.

1.2. *A finite-orbit obstruction.* Write ℓ_C and A_C for the Euclidean perimeter and enclosed area of a curve C . Balancing the tangent distribution of one curve gives a normalization in which the failure of the classification is measured by a product of isoperimetric ratios.

THEOREM 1.2 (Finite variational obstruction). *Let Q, P be any smooth positively curved convex pair. Choose $A \in \mathrm{SL}(2, \mathbb{R})$ by minimizing $\mathrm{Per}(AQ)$, as in the proof of Lemma 4.1. Apply A to both curves and relabel the transformed pair as (Q, P) , with perimeters and areas ℓ_Q, ℓ_P, A_Q, A_P . Set*

$$\Gamma = \frac{\ell_Q \ell_P}{4\pi \sqrt{A_Q A_P}} \geq 1.$$

If $\Gamma > 1$, then for every integer

$$(1) \quad m > \frac{\Gamma}{\Gamma - 1}$$

there is a nonempty open set of physical orbit segments of (Q, P) , with m internal vertices, whose fixed-endpoint Hessians have a positive direction. The same holds for the original pair before normalization. Moreover, $\Gamma = 1$ if and only if the original pair belongs to the class in Theorem 1.1.

The bound is for the number of internal alternating vertices; a segment has $m + 2$ vertices in total. Its conclusion concerns an open set of initial conditions, not merely an exceptional orbit. No claim of a sharp threshold, positive entropy, or a Lyapunov exponent is needed.

Theorem 1.2 is proved using a variation that is constant at the internal vertices. Proposition 4.3 gives the same positive-direction conclusion under the weaker condition

$$\Gamma \cos \frac{\pi}{m+1} > 1,$$

by using the first discrete Dirichlet sine mode. The resulting sufficient length grows as $(\Gamma - 1)^{-1/2}$ near the equality class; see Remark 4.4.

1.3. *Minkowski applications.* Let B_2 denote the closed Euclidean unit disk. For a smooth convex body K containing 0 in its interior, its gauge is

$$N_K(x) = \inf\{r > 0 : x \in rK\}.$$

It is not assumed to be reversible. The billiard in a smooth convex table D , with boundary parametrization q , has generating function

$$(2) \quad G(s, u) = N_K(q(u) - q(s)).$$

Here total integrability means a full foliation of the non-grazing phase annulus by continuous invariant essential graphs over the boundary coordinate. Definition 5.1 gives this notion before any strict curvature assumption on the table.

THEOREM 1.3 (Metric-table rigidity). *Let K be a smooth convex unit ball containing 0 in its interior, with everywhere positive curvature. Let D be a compact convex body with nonempty interior and smooth boundary, used as the billiard table. The billiard in D with gauge N_K is totally integrable if and only if there exist an orientation-preserving linear isomorphism B , a vector $\beta \in \mathbb{R}^2$ with $|\beta| < 1$, a center $c \in \mathbb{R}^2$, and $R > 0$ such that*

$$(3) \quad N_K(x) = |Bx| + \beta \cdot Bx, \quad BD = c + RB_2.$$

The linear term in this statement cannot be omitted. Its contribution to an edge is the difference of its values at the two endpoints, so it does not change the reflection law or a fixed-endpoint Hessian. The self condition $D = K$ removes this freedom. In that case positive curvature of K follows from total integrability, rather than being an additional assumption.

COROLLARY 1.4 (Self-Minkowski rigidity). *Let $K \subset \mathbb{R}^2$ be a compact convex body containing 0 in its interior, with C^∞ boundary, and let*

$$N_K(x) = \inf\{r > 0 : x \in rK\}$$

be its possibly nonreversible gauge. Suppose that the self-Minkowski billiard map in K is totally integrable on its entire non-grazing phase annulus, in the sense of Definition 5.1. Then there exists a positive-definite symmetric matrix H such that

$$(4) \quad N_K(x) = \sqrt{x^\top H x} \quad (x \in \mathbb{R}^2).$$

Equivalently, K is an ellipse centered at the origin. Conversely, every gauge of the form (4) has a totally integrable self-Minkowski billiard.

This answers Question 2 in the contribution of Misha Bialy to [2, pp. 1–2], communicated by Yaron Ostrover. The question explicitly allows asymmetric norms and uses a full invariant-graph foliation. The same problem is recorded in [9, §9.5]. The curvature distinction between Theorem 1.3 and Corollary 1.4 is essential to the statements: the former assumes a strongly convex metric, whereas the latter starts only with a smooth convex unit ball.

1.4. *The averaging mechanism.* Let X be the disjoint union of the two physical phase components, with its finite invariant area measure μ . For the alternating edge Lagrangian L_ϵ , put

$$b = (L_\epsilon)_{12} > 0, \quad h = (L_\epsilon)_{11} + 2(L_\epsilon)_{12} + (L_\epsilon)_{22}.$$

For an orbit segment with m internal vertices, let $\mathcal{Q}_m$ be the second variation with value 1 at every internal vertex and 0 at both endpoints. Invariance of μ gives the exact identity

$$(5) \quad \int_X \mathcal{Q}_m \, d\mu = m\mathcal{I} - 2\mathcal{B}, \quad \mathcal{I} = \int_X h \, d\mu, \quad \mathcal{B} = \int_X b \, d\mu > 0.$$

It holds for every positively curved pair, without a variational sign assumption. Under the hypothesis of the pair theorem, division by m and passage to the limit give $\mathcal{I} \leq 0$. No infinite-orbit limit is taken inside an integral.

If $v_Q, v_P \in \mathbb{R}^2$ denote the second tangent moments defined in (30)–(31), direct integration gives

$$(6) \quad \mathcal{I} = \ell_Q \ell_P - v_Q \cdot v_P - 2\pi(A_Q + A_P), \quad 2\mathcal{B} = \ell_Q \ell_P - v_Q \cdot v_P.$$

A common area-preserving linear map can be chosen so that $v_Q = 0$. Applying the resulting inequality to $(Q, \lambda P)$, for every $\lambda > 0$, gives

$$\lambda \ell_Q \ell_P \leq 2\pi(A_Q + \lambda^2 A_P).$$

Minimization in λ yields $\ell_Q \ell_P \leq 4\pi \sqrt{A_Q A_P}$. The two Euclidean isoperimetric inequalities force equality for both curves, so both normalized curves are circles. Keeping the finite boundary term in (5) gives Theorem 1.2. The weighted averaging identity (55) and its discrete Dirichlet optimization yield the shorter sufficient length in Proposition 4.3.

For the Minkowski applications, the symplectic polar satisfies

$$N_K(v) = \max_{p \in K^\omega} \omega(p, v).$$

Eliminating the polar vertices from the alternating action of $(\partial D, \partial K^\omega)$ therefore gives exactly the Minkowski generating function. Continuous invariant graphs calibrate the reduced action; a Schur-complement identity transfers its

Hessian sign to the full alternating action. This proves Theorem 1.3. In the self case, a zero-curvature point would force the gauge Hessian to vanish on an inward half-plane, contradicting positivity of the gauge in opposite directions. Finally, a translated disk has a circular polar only when its center is the origin. These two facts give Corollary 1.4.

1.5. *Previous work and organization.* Bialy’s billiard version of Hopf rigidity [8] and the discrete variational theory of Bialy and MacKay [11] provide the background for the sign-sensitive second-variation argument. Symplectic billiards were introduced by Albers and Tabachnikov [5]. Baracco and Bernardi proved rigidity for totally integrable one-curve symplectic billiards and used affine normalization to eliminate the second Fourier terms [7]. The present two-curve proof retains that normalization principle and uses independent dilation to force equality for two curves simultaneously. The finite-block identity supplies both the Hopf inequality and its finite-orbit counterpart.

The pair construction is related to the systems of Albers, Lander, and Westermann [4]. The reduction to Minkowski billiards is related to the symplectic description and square-root correspondence of Albers, Chávez Cáliz, and Tabachnikov [3]. We prove the global phase correspondence and the Hessian identity needed here explicitly. Minkowski billiards were developed by Gutkin and Tabachnikov [12] and also occur in the symplectic approach to Mahler-type questions [6]. Curvature obstructions for Minkowski caustics appear in [12, §6]; our curvature argument works directly with continuous invariant graphs before strong convexity of the self-gauge is known.

Bialy and Tsodikovich proved self-Minkowski rigidity under invariance by a linear map of finite order at least three [10, Theorem 5]. Their theorem assumes invariance under a finite-order linear map of order at least three; no symmetry assumption is made here. For independent metric–table pairs, the exact linear freedom is consistent with the equivalence of billiard reflection laws studied in [1].

The self-Minkowski corollary also resolves the Minkowski-billiard problem posed as Question 2 in the open-problem list of Albach et al. [2, Question 2, p. 2], in the contribution of Misha Bialy and communicated by Yaron Ostrover. That question asks whether, for a Minkowski norm N , not assumed symmetric, total integrability of the billiard in the unit ball of N forces N to be Euclidean. Corollary 1.4 gives an affirmative answer in the smooth convex billiard-map formulation stated there: no reversibility or finite-symmetry hypothesis is imposed, and positive curvature is derived from total integrability rather than assumed. In particular, this removes the finite-symmetry hypothesis from the earlier affirmative result of Bialy and Tsodikovich [10, Theorem 5].

Sections 2–4 prove the two general theorems without using Minkowski billiards. Section 5 establishes calibration and automatic curvature for a smooth convex table. Section 6 proves the polar reduction and the variational transfer. Sections 7 and 8 prove the metric–table theorem and its self-Minkowski corollary. Appendix A gives an independent Green–Riccati proof of the Hopf inequality, and Appendix B records the gauge of a translated disk.

2. The alternating action on two convex curves

2.1. *Orientation and physical chords.* We fix

$$(7) \quad \omega(x, y) = \det(x, y), \quad J(x_1, x_2) = (-x_2, x_1),$$

so that $\omega(x, y) = \langle Jx, y \rangle$. All convex curves are positively oriented, that is, counterclockwise. If c is parametrized by Euclidean arclength and $T = c'$, then

$$(8) \quad c'' = \kappa JT, \quad \kappa \geq 0,$$

where JT is the inward unit normal. In this and the next two sections, both curves have positive curvature. The weak inequality will be used only in the later Minkowski applications. In particular,

$$(9) \quad \omega(c'', c') = -\kappa, \quad \int_C \kappa \, ds = 2\pi.$$

LEMMA 2.1 (Chord orientation). *Let $c : \mathbb{R}/\ell\mathbb{Z} \rightarrow C$ be a positively oriented regular C^1 parametrization of a strictly convex closed curve. For lifts $a < b < a + \ell$,*

$$(10) \quad \omega(c'(a), c(b) - c(a)) > 0, \quad \omega(c(b) - c(a), c'(b)) > 0.$$

Consequently, a nonzero vector d based at $c(a)$ is inward-pointing, meaning that $c(a) + \rho d$ lies in the interior for all sufficiently small $\rho > 0$, if and only if $\omega(c'(a), d) > 0$.

Proof. The interior of a positively oriented convex curve lies in the closed left half-plane determined by every oriented tangent line. Strict convexity makes the inclusion strict away from the tangency point. Applying this at $c(a)$ gives the first inequality. Applying it at $c(b)$ to the chord $c(a) - c(b)$ gives $\omega(c'(b), c(a) - c(b)) > 0$, which is the second inequality after antisymmetry of ω . $\square$

Let Q and P be any smooth closed convex curves of positive curvature. Parametrize them counterclockwise by Euclidean arclength,

$$q : \mathbb{R}/\ell_Q\mathbb{Z} \longrightarrow Q, \quad p : \mathbb{R}/\ell_P\mathbb{Z} \longrightarrow P.$$

Define

$$(11) \quad L_0(s, t) = \omega(q(s), p(t)), \quad L_1(t, u) = \omega(p(t), q(u)).$$

The relevant physical phase cylinders are

$$(12) \quad \mathcal{P}_0 = \{(s, t) : \omega(q'(s), p'(t)) > 0\},$$

$$(13) \quad \mathcal{P}_1 = \{(t, u) : \omega(p'(t), q'(u)) > 0\}.$$

All parameter pairs are understood modulo their respective periods.

2.2. *The alternating map.* For $(s, t) \in \mathcal{P}_0$, Lemma 2.1 says that the ray

$$q(s) + \rho p'(t), \quad \rho > 0,$$

points into the convex body bounded by Q . Strict convexity implies that it meets Q at exactly one further point $q(u)$, and

$$(14) \quad q(u) - q(s) = \rho p'(t), \quad \rho > 0.$$

The second inequality in (10) gives $\omega(p'(t), q'(u)) > 0$, so $(t, u) \in \mathcal{P}_1$. Likewise, from $(t, u) \in \mathcal{P}_1$ there is a unique v such that

$$(15) \quad p(v) - p(t) = \sigma q'(u), \quad \sigma > 0,$$

and $(u, v) \in \mathcal{P}_0$. These constructions define smooth diffeomorphisms

$$(16) \quad T_0 : \mathcal{P}_0 \longrightarrow \mathcal{P}_1, \quad T_1 : \mathcal{P}_1 \longrightarrow \mathcal{P}_0.$$

The one-step map on the disjoint union alternates between these components; its square restricts to a map of each component. We use the one-step convention throughout the averaging argument.

The same map follows from the variational principle for (11). Indeed, stationarity at a P -vertex gives

$$\begin{aligned} 0 &= (L_0)_2(s, t) + (L_1)_1(t, u) \\ &= \omega(q(s), p'(t)) + \omega(p'(t), q(u)) \\ &= \omega(p'(t), q(u) - q(s)), \end{aligned}$$

which is (14). Stationarity at a Q -vertex gives

$$\begin{aligned} 0 &= (L_1)_2(t, u) + (L_0)_1(u, v) \\ &= \omega(p(t), q'(u)) + \omega(q'(u), p(v)) \\ &= \omega(p(t) - p(v), q'(u)), \end{aligned}$$

which is (15). Thus an orbit is an alternating critical configuration

$$(17) \quad \dots, s_i, t_i, s_{i+1}, t_{i+1}, \dots$$

for the action obtained by summing L_0 and L_1 .

2.3. *Finite Hessians and endpoint extension.* Write

$$X = \mathcal{P}_0 \sqcup \mathcal{P}_1, \quad \mathcal{T}|_{\mathcal{P}_0} = T_0, \quad \mathcal{T}|_{\mathcal{P}_1} = T_1.$$

All coordinates are Euclidean arclength. On an orbit, let x_n denote the vertex parameter and let $\mathcal{L}_n$ be the generating function on the edge (x_n, x_{n+1}) . Depending on the initial component, the even vertices lie on either Q or P ; this does not affect any formula. Set

$$(18) \quad b_n = (\mathcal{L}_n)_{12}(x_n, x_{n+1}) > 0,$$

$$(19) \quad a_n = (\mathcal{L}_{n-1})_{22}(x_{n-1}, x_n) + (\mathcal{L}_n)_{11}(x_n, x_{n+1}).$$

The fixed-endpoint Hessian has diagonal a_n and off-diagonal b_n . We call nonpositivity of all these finite Hessians *second-variation maximality*; its strict version means negative definiteness. These terms do not assert a uniform spectral gap.

LEMMA 2.2 (Strictness by endpoint extension). *Suppose that every finite fixed-endpoint Hessian along a bi-infinite scalar nearest-neighbor orbit is negative semidefinite, and that all off-diagonal coefficients are nonzero. Every such Hessian with an internal vertex is then negative definite. The same statement holds with both signs reversed.*

Proof. Let H be the Hessian on internal vertices $x_1, \dots, x_m$ and suppose that $H\xi = 0$ for a nonzero vector ξ . Extend the segment by one vertex at its right endpoint and extend the vector by zero at the former endpoint. The enlarged Hessian $\widehat{H}$ is negative semidefinite and

$$(\xi, 0)^\top \widehat{H}(\xi, 0) = \xi^\top H\xi = 0.$$

For a symmetric semidefinite matrix, a vector with zero quadratic form lies in its kernel. The last row of $\widehat{H}(\xi, 0) = 0$ is $b_m \xi_m = 0$. Since $b_m \neq 0$, it gives $\xi_m = 0$. The preceding tridiagonal rows successively give $\xi_{m-1} = \dots = \xi_1 = 0$, a contradiction. The positive version follows by multiplying every Hessian by -1 . $\square$

At a critical configuration, reparametrizing the internal vertices changes the Hessian by an invertible congruence. Independent translations of Q and P also preserve every fixed-endpoint Hessian: at each internal vertex the new linear terms cancel because its two neighbors lie on the same other curve. Only endpoint terms and constants remain. These observations allow different centers and arclength parametrizations throughout the pair theorem.

2.4. *The finite invariant measure.* The natural area forms are

$$(20) \quad d\mu_0 = (L_0)_{12}(s, t) ds dt = \omega(q'(s), p'(t)) ds dt \quad \text{on } \mathcal{P}_0,$$

$$(21) \quad d\mu_1 = (L_1)_{12}(t, u) dt du = \omega(p'(t), q'(u)) dt du \quad \text{on } \mathcal{P}_1.$$

They are positive by definition of the physical phases.

LEMMA 2.3 (Alternating invariant area). *The alternating map preserves the measure on $\mathcal{P}_0 \sqcup \mathcal{P}_1$ defined by (20)–(21): T_0 sends μ_0 to μ_1 , and T_1 sends μ_1 to μ_0 .*

Proof. Along any edge sequence, the Euler–Lagrange equation at x_{n+1} is

$$(\mathcal{L}_n)_2(x_n, x_{n+1}) + (\mathcal{L}_{n+1})_1(x_{n+1}, x_{n+2}) = 0.$$

Differentiating gives

$$b_n dx_n + ((\mathcal{L}_n)_{22} + (\mathcal{L}_{n+1})_{11}) dx_{n+1} + b_{n+1} dx_{n+2} = 0.$$

Wedge this identity with dx_{n+1} . The middle term vanishes and one obtains

$$b_n dx_n \wedge dx_{n+1} = b_{n+1} dx_{n+1} \wedge dx_{n+2}.$$

This is exactly the asserted invariance. $\square$

Since the tangents have unit length, $0 < b \leq 1$ on X . Therefore

$$(22) \quad 0 < \mu(X) \leq 2\ell_Q \ell_P < \infty.$$

All pure derivatives of L_0, L_1 are bounded on their compact parameter tori. Thus all functions integrated below are integrable. The density is strictly positive at every physical point, so every nonempty open subset of X has positive measure.

3. Finite-block averaging and the geometric identity

The argument begins with an identity valid without any variational sign condition. The two components must be averaged together; their contributions combine into one integral on the full product torus.

3.1. *The finite-block identity.* Define on an edge of type $\epsilon \in \{0, 1\}$

$$b = (L_\epsilon)_{12}, \quad h = (L_\epsilon)_{11} + 2(L_\epsilon)_{12} + (L_\epsilon)_{22},$$

and put

$$(23) \quad \mathcal{I} = \int_X h d\mu, \quad \mathcal{B} = \int_X b d\mu > 0.$$

For an integer $m \geq 1$, consider the orbit segment $x_0, \dots, x_{m+1}$ issued from $z = (x_0, x_1) \in X$ and the variation

$$\xi_0 = \xi_{m+1} = 0, \quad \xi_j = 1 \quad (1 \leq j \leq m).$$

Denote its second variation by $\mathcal{Q}_m(z)$. This is a genuine fixed-endpoint variation: for each finite segment, a sufficiently small multiple of ξ preserves all strict physical phase inequalities.

PROPOSITION 3.1 (Finite-block averaging identity). *For every smooth positively curved pair, without any maximality assumption, one has*

$$(24) \quad \int_X \mathcal{Q}_m(z) d\mu(z) = m\mathcal{I} - 2\mathcal{B} \quad (m \geq 1).$$

Proof. The tridiagonal Hessian gives

$$\mathcal{Q}_m(z) = \sum_{j=1}^m a_j(z) + 2 \sum_{j=1}^{m-1} b_j(z).$$

Write d_{11}, d_{22} for the pure derivatives on an edge, according to its type. Then

$$a_j(z) = d_{22}(\mathcal{T}^{j-1}z) + d_{11}(\mathcal{T}^jz), \quad b_j(z) = b(\mathcal{T}^jz).$$

Measure invariance, with no ergodicity assumption, yields

$$\begin{aligned} \int_X \mathcal{Q}_m d\mu &= m \int_X (d_{11} + d_{22}) d\mu + 2(m-1) \int_X b d\mu \\ &= m \int_X h d\mu - 2 \int_X b d\mu. \end{aligned}$$

This is (24). □

PROPOSITION 3.2 (Alternating Hopf inequality). *If every finite physical alternating segment has a negative-semidefinite fixed-endpoint Hessian, then*

$$(25) \quad \begin{aligned} &\int_{\mathcal{P}_0} ((L_0)_{11} + 2(L_0)_{12} + (L_0)_{22}) d\mu_0 \\ &+ \int_{\mathcal{P}_1} ((L_1)_{11} + 2(L_1)_{12} + (L_1)_{22}) d\mu_1 \leq 0. \end{aligned}$$

For

$$(26) \quad L(s, t) = \omega(q(s), p(t)),$$

this is equivalently

$$(27) \quad \int_0^{\ell_Q} \int_0^{\ell_P} (L_{ss} + 2L_{st} + L_{tt}) L_{st} dt ds \leq 0.$$

Proof. Under the hypothesis, $\mathcal{Q}_m(z) \leq 0$ for every z and every $m \geq 1$. Proposition 3.1 gives

$$m\mathcal{I} \leq 2\mathcal{B} \quad (m \geq 1).$$

Divide by m and let $m \rightarrow \infty$. Since $\mathcal{B}$ is finite, $\mathcal{I} \leq 0$, which proves (25). This limit is a limit of scalar inequalities; no interchange of an infinite-orbit limit and an integral is involved.

To pass to the full parameter torus, put

$$L(s, t) = \omega(q(s), p(t)), \quad \mathcal{J}(s, t) = (L_{ss} + 2L_{st} + L_{tt}) L_{st}.$$

The first physical component is the region $L_{st} > 0$, and its contribution is the integral of $\mathcal{J}$ over that region.

For the second component,

$$L_1(t, s) = -L(s, t).$$

After exchanging the two coordinates,

$$(L_1)_{11} = -L_{tt}, \quad (L_1)_{12} = -L_{st}, \quad (L_1)_{22} = -L_{ss}.$$

The density is the positive measure

$$d\mu_1 = (-L_{st}) dt ds$$

on the region $L_{st} < 0$. Here the coordinate exchange is a change of variables for positive measures, with absolute Jacobian equal to one; it does not introduce an additional orientation sign. Its contribution is therefore

$$\int_{\{L_{st} < 0\}} [-(L_{ss} + 2L_{st} + L_{tt})] (-L_{st}) ds dt = \int_{\{L_{st} < 0\}} \mathcal{J}(s, t) ds dt.$$

On the remaining set $L_{st} = 0$, the integrand $\mathcal{J}$ vanishes identically. Thus the two physical integrals combine into the full-torus integral. No measure-zero assertion about the parallel-tangent set is needed. $\square$

Remark 3.3. In particular, the main rigidity argument does not require an invariant Green field, a uniform bound away from grazing, or a measurable choice of invariant leaf. Appendix A gives an independent Green–Riccati proof under the strict second-variation hypothesis, independently of any invariant-graph foliation.

3.2. Evaluation on the product torus. Let $A_Q, A_P > 0$ be the Euclidean areas enclosed by Q, P , and let ℓ_Q, ℓ_P be their Euclidean perimeters. We first record a bilinear form of Green’s theorem.

LEMMA 3.4 (Bilinear area identity). *Let C be a positively oriented regular C^1 simple closed curve enclosing area A_C . For every constant $a, b \in \mathbb{R}^2$,*

$$(28) \quad \oint_C \omega(a, x) \omega(b, dx) = A_C \omega(a, b).$$

Proof. The left-hand side is bilinear in (a, b) . Interchanging a and b and adding gives

$$\oint_C d(\omega(a, x) \omega(b, x)) = 0,$$

so it is alternating. Every alternating bilinear form on $\mathbb{R}^2$ is a scalar multiple of ω . Taking $a = e_1$ and $b = e_2$ gives

$$\oint_C \omega(e_1, x) \omega(e_2, dx) = - \oint_C x_2 dx_1 = A_C$$

by Green's theorem. Since $\omega(e_1, e_2) = 1$, the scalar is A_C . $\square$

Write the unit tangents as

$$(29) \quad q'(s) = (\cos \alpha(s), \sin \alpha(s)), \quad p'(t) = (\cos \beta(t), \sin \beta(t)).$$

Define the second tangent moments

$$(30) \quad v_Q = \left(\int_0^{\ell_Q} \cos(2\alpha(s)) ds, \int_0^{\ell_Q} \sin(2\alpha(s)) ds \right),$$

$$(31) \quad v_P = \left(\int_0^{\ell_P} \cos(2\beta(t)) dt, \int_0^{\ell_P} \sin(2\beta(t)) dt \right).$$

PROPOSITION 3.5 (Exact two-curve formula). *For $L(s, t) = \omega(q(s), p(t))$,*

$$(32) \quad \begin{aligned} \int_0^{\ell_Q} \int_0^{\ell_P} (L_{ss} + 2L_{st} + L_{tt}) L_{st} dt ds \\ = \ell_Q \ell_P - v_Q \cdot v_P - 2\pi(A_Q + A_P). \end{aligned}$$

Consequently, second-variation maximality of all finite alternating segments implies

$$(33) \quad \ell_Q \ell_P - v_Q \cdot v_P \leq 2\pi(A_Q + A_P).$$

Proof. We evaluate the three terms in (27). First, for fixed s , Lemma 3.4 applied to P with $a = q''(s)$ and $b = q'(s)$ gives

$$\begin{aligned} \int_0^{\ell_P} L_{ss} L_{st} dt &= \int_0^{\ell_P} \omega(q''(s), p(t)) \omega(q'(s), p'(t)) dt \\ &= A_P \omega(q''(s), q'(s)). \end{aligned}$$

Using (9) and integrating in s ,

$$(34) \quad \int_0^{\ell_Q} \int_0^{\ell_P} L_{ss} L_{st} dt ds = -2\pi A_P.$$

For the second pure term, antisymmetry of ω gives

$$\omega(q, p'') \omega(q', p') = \omega(p'', q) \omega(p', q').$$

For fixed t , Lemma 3.4 applied to Q with $a = p''(t)$ and $b = p'(t)$ therefore yields

$$\int_0^{\ell_Q} L_{tt} L_{st} ds = A_Q \omega(p''(t), p'(t)).$$

A second use of (9) gives

$$(35) \quad \int_0^{\ell_Q} \int_0^{\ell_P} L_{tt} L_{st} dt ds = -2\pi A_Q.$$

Finally,

$$L_{st} = \omega(q'(s), p'(t)) = \sin(\beta(t) - \alpha(s)).$$

Hence

$$(36) \quad \begin{aligned} 2 \int_0^{\ell_Q} \int_0^{\ell_P} L_{st}^2 dt ds &= \int_0^{\ell_Q} \int_0^{\ell_P} [1 - \cos(2\beta(t) - 2\alpha(s))] dt ds \\ &= \ell_Q \ell_P - v_Q \cdot v_P. \end{aligned}$$

Adding (34), (35), and (36) proves (32). Inequality (33) follows from Proposition 3.2. $\square$

COROLLARY 3.6 (The averaged boundary term). *For every smooth positively curved pair,*

$$(37) \quad 2\mathcal{B} = \ell_Q \ell_P - v_Q \cdot v_P.$$

Consequently the finite-block identity is

$$(38) \quad \int_X \mathcal{Q}_m d\mu = m(\ell_Q \ell_P - v_Q \cdot v_P - 2\pi(A_Q + A_P)) - (\ell_Q \ell_P - v_Q \cdot v_P).$$

Proof. On $\mathcal{P}_0$, $b d\mu = L_{st}^2 ds dt$; on the swapped second component, $b d\mu = (-L_{st})^2 ds dt$. The integrand vanishes on $L_{st} = 0$. Thus

$$\mathcal{B} = \int_0^{\ell_Q} \int_0^{\ell_P} L_{st}^2 dt ds.$$

Equation (36) gives (37); substituting this and Proposition 3.5 into (24) gives (38). $\square$

4. Affine rigidity and the finite-orbit obstruction

4.1. *Balancing the second tangent moment.* The alternating action is invariant under a common area-preserving linear map. Indeed, in dimension two,

$$(39) \quad A \in \mathrm{SL}(2, \mathbb{R}) \iff \omega(Ax, Ay) = \omega(x, y) \iff A \in \mathrm{Sp}(2, \mathbb{R}).$$

Thus the correspondence of orbit segments under $(Q, P) \mapsto (AQ, AP)$ preserves the alternating action exactly. Reparametrizing the transformed curves changes a critical Hessian by congruence, so strict second-variation maximality is also preserved.

LEMMA 4.1 (Area-preserving tangent balancing). *For every smooth strictly convex closed curve Q , there exists $A \in \text{SL}(2, \mathbb{R})$ such that*

$$(40) \quad v_{AQ} = 0,$$

where the tangent moment is computed after parametrizing AQ by Euclidean arclength.

Proof. Consider the Euclidean perimeter functional

$$(41) \quad \mathcal{L}(A) = \text{Per}(AQ), \quad A \in \text{SL}(2, \mathbb{R}).$$

We first show that it attains a minimum modulo left rotations. By polar decomposition, $A = RS$ with $R \in \text{SO}(2)$ and S positive definite with determinant one. Left multiplication by R does not change perimeter. Let the singular values of S be λ and λ^{-1} , with $\lambda \geq 1$, and let e be a unit eigenvector of S for λ . If $w_Q(\theta)$ denotes the width of Q in direction θ and

$$w_{\min}(Q) = \min_{|\theta|=1} w_Q(\theta) > 0,$$

then the width of SQ in direction e is

$$w_{SQ}(e) = w_Q(S^T e) = \lambda w_Q(e) \geq \lambda w_{\min}(Q).$$

The length of a closed curve is at least the total variation of each of its one-dimensional projections. That total variation is at least twice the width in the projection direction. Consequently,

$$(42) \quad \text{Per}(SQ) \geq 2\lambda w_{\min}(Q),$$

which tends to infinity as $\lambda \rightarrow \infty$.

The positive determinant-one matrices with bounded singular values form a compact set. Thus (41) attains a minimum at some A_0 , after ignoring an irrelevant left rotation.

Put $C = A_0 Q$ and parametrize C by Euclidean arclength. Write its unit tangent as $T(s)$. For every symmetric trace-free matrix B , the path $e^{\varepsilon B}$ lies in $\text{SL}(2, \mathbb{R})$, and minimality gives

$$(43) \quad 0 = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \text{Per}(e^{\varepsilon B} C) = \int_C \langle BT, T \rangle ds.$$

Set

$$M_C = \int_C TT^T ds.$$

Equation (43) says that the symmetric matrix M_C is orthogonal, under the trace pairing, to every symmetric trace-free matrix. Hence

$$(44) \quad M_C = \frac{\ell_C}{2} I,$$

because $\text{tr } M_C = \int_C |T|^2 ds = \ell_C$. If $T = (\cos \alpha, \sin \alpha)$, then

$$(45) \quad M_C - \frac{\ell_C}{2} I = \frac{1}{2} \begin{pmatrix} \int_C \cos 2\alpha ds & \int_C \sin 2\alpha ds \\ \int_C \sin 2\alpha ds & -\int_C \cos 2\alpha ds \end{pmatrix}.$$

Thus (44) is exactly $v_C = 0$. Taking $A = A_0$ proves the lemma. $\square$

4.2. Independent scaling of one curve. The next simple freedom is decisive. It is unavailable for a one-curve symplectic billiard but available for an alternating pair.

LEMMA 4.2 (One-sided scaling). *If all finite fixed-endpoint Hessians for (Q, P) are negative semidefinite, respectively negative definite, the same is true for $(Q, \lambda P)$ for every $\lambda > 0$.*

Proof. For $\lambda > 0$, the correspondence

$$q_i \mapsto q_i, \quad p_i \mapsto \lambda p_i$$

is a bijection between physical alternating orbit configurations for (Q, P) and for $(Q, \lambda P)$. Indeed, tangent orientations are unchanged, and

$$q_{i+1} - q_i \parallel p'(t_i), \quad \lambda p_{i+1} - \lambda p_i \parallel q'(s_{i+1})$$

are exactly the corresponding chord conditions. Their physical signs are preserved because $\lambda > 0$.

Using the original parameters on both curves, every generating term, and therefore the whole action, is multiplied by λ :

$$\omega(q, \lambda p) = \lambda \omega(q, p), \quad \omega(\lambda p, q) = \lambda \omega(p, q).$$

Thus the fixed-endpoint Hessian in those parameters is multiplied by λ .

To use Euclidean arclength on the scaled curve, write

$$\tilde{p}(\tilde{t}) = \lambda p(\tilde{t}/\lambda).$$

If C is the diagonal Jacobian of the change from the new internal parameters to the old ones, its entries are 1 at Q -vertices and $1/\lambda$ at P -vertices. At a critical configuration,

$$\text{Hess } \tilde{\mathcal{A}} = \lambda C^T (\text{Hess } \mathcal{A}) C.$$

There are no additional first-derivative terms because the action is critical in all internal variables. Since C is invertible and $\lambda > 0$, semidefiniteness and definiteness are both preserved. $\square$

We can now prove the pair theorem announced in the introduction.

Proof of Theorem 1.1. Lemma 2.2 gives (i) $\Rightarrow$ (ii), since the physical coefficients b_n are positive. We prove (i) $\Rightarrow$ (iii) directly, to keep the geometric argument under its weakest sign hypothesis.

Use Lemma 4.1 to choose $A \in \text{SL}(2, \mathbb{R})$ such that $v_{AQ} = 0$, and apply A simultaneously to both curves. By (39), second-variation maximality is preserved. Relabel the transformed pair as (Q, P) , so

$$(46) \quad v_Q = 0.$$

The scaling argument takes place in the class of arbitrary convex pairs. The scaled pair need not remain a polar pair: polarity is used only later, in Section 6. For every $\lambda > 0$, Lemma 4.2 permits us to apply (33) to $(Q, \lambda P)$. Under this scaling,

$$(47) \quad \ell_{\lambda P} = \lambda \ell_P, \quad A_{\lambda P} = \lambda^2 A_P, \quad v_{\lambda P} = \lambda v_P.$$

Together with (46), inequality (33) becomes

$$(48) \quad \lambda \ell_Q \ell_P \leq 2\pi(A_Q + \lambda^2 A_P).$$

Divide by λ and minimize the right-hand side over $\lambda > 0$. The minimum occurs at $\lambda = \sqrt{A_Q/A_P}$ and gives

$$(49) \quad \ell_Q \ell_P \leq 4\pi \sqrt{A_Q A_P}.$$

On the other hand, the Euclidean isoperimetric inequality applied separately to Q and P gives

$$(50) \quad \ell_Q \geq 2\sqrt{\pi A_Q}, \quad \ell_P \geq 2\sqrt{\pi A_P},$$

and hence the reverse inequality to (49). Equality follows throughout. Since both ratios

$$\frac{\ell_Q}{2\sqrt{\pi A_Q}}, \quad \frac{\ell_P}{2\sqrt{\pi A_P}}$$

are at least one and their product is one, each ratio equals one. Equality in the planar isoperimetric inequality forces each curve to be a Euclidean circle, with no restriction on its center. Thus, in the balanced coordinates,

$$Q = c_Q + r_Q S^1, \quad P = c_P + r_P S^1.$$

Undoing A proves (iii).

It remains to prove (iii) implies (ii). Common area-preserving linear changes preserve critical Hessians by congruence. Independent translations of the curves change a fixed-endpoint action only by endpoint terms and constants: at an internal vertex x_j , the added terms cancel because the adjacent vertices lie on the same other curve. Thus we may assume

$$q(\theta) = r_Q(\cos \theta, \sin \theta), \quad p(\phi) = r_P(\cos \phi, \sin \phi).$$

The angular increments of consecutive vertices of every physical orbit are equal to a number $\delta \in (0, \pi)$. To verify this, the chord between angles θ and $\theta + 2\delta$ on the first circle is a positive multiple of the tangent at angle $\theta + \delta$ on the second; the same construction with the circles exchanged propagates the increment δ .

In angular coordinates the action, up to the irrelevant endpoint terms, is

$$r_Q r_P \sum_j \sin(\theta_{j+1} - \theta_j).$$

Its second variation along this orbit is

$$(51) \quad -r_Q r_P \sin \delta \sum_j (\xi_{j+1} - \xi_j)^2.$$

For a nonzero variation with both endpoints zero, at least one difference is nonzero. Since $\sin \delta > 0$, the expression is strictly negative. Reparametrization to Euclidean arclength is an invertible coordinate change, so it preserves definiteness at the critical configuration. This proves (ii), and (ii) implies (i) immediately. $\square$

4.3. *A finite-length obstruction outside the equality class.* The preceding proof yields more than the failure of global maximality. It bounds the length at which a positive second variation must occur.

Proof of Theorem 1.2. Set $\lambda = \sqrt{A_Q/A_P}$ and apply the finite-block identity to $(Q, \lambda P)$ in its Euclidean arclength parameters. The orbit correspondence and the congruence in Lemma 4.2 hold without any sign assumption. For the scaled pair,

$$v_Q = 0, \quad A_{\lambda P} = A_Q, \quad \ell_Q \ell_{\lambda P} = 4\pi A_Q \Gamma.$$

Equations (32) and (37) give

$$\mathcal{I}_\lambda = 4\pi A_Q (\Gamma - 1), \quad 2\mathcal{B}_\lambda = 4\pi A_Q \Gamma.$$

Consequently,

$$(52) \quad \int_{X_\lambda} \mathcal{Q}_m d\mu_\lambda = 4\pi A_Q [m(\Gamma - 1) - \Gamma] > 0.$$

The continuous function $\mathcal{Q}_m$ is therefore positive somewhere. Its positive set is nonempty and open, and has positive physical measure. At each of these points the constant-internal-coordinate variation is an explicit positive direction of the Hessian. The orbit correspondence and invertible Hessian congruences transport this conclusion back through the scaling and the common linear map. They also transport openness.

If $\Gamma = 1$, the two individual isoperimetric ratios are at least one and have product one, so both normalized curves are circles. Conversely, for a common-shape elliptic pair, a determinant-one map makes Q a circle. Its perimeter is then the isoperimetric minimum; every perimeter minimizer therefore also makes Q a circle. Homothety makes the other curve a circle under that same map, and hence $\Gamma = 1$. $\square$

The constant internal variation used above is not optimal for the averaged quadratic form. Allowing nonconstant weights gives the following refinement.

PROPOSITION 4.3 (Discrete-sine refinement). *Use the hypotheses and the balanced normalization of Theorem 1.2, and suppose that $\Gamma > 1$. For every integer $m \geq 1$ satisfying*

$$(53) \quad \Gamma \cos \frac{\pi}{m+1} > 1,$$

or, equivalently,

$$m+1 > \frac{\pi}{\arccos(\Gamma^{-1})},$$

there is a nonempty open set of physical initial conditions whose alternating orbit segments with m internal vertices have a fixed-endpoint Hessian with a positive direction. This set has positive physical measure. The same conclusion holds for the original pair before normalization.

After the additional scaling $(Q, P) \mapsto (Q, \lambda P)$ with $\lambda = \sqrt{A_Q/A_P}$, an explicit positive direction on a nonempty open set is given, in Euclidean arclength coordinates, by

$$(54) \quad \xi_0 = \xi_{m+1} = 0, \quad \xi_j = \sin \frac{\pi j}{m+1} \quad (1 \leq j \leq m).$$

Proof. We first record the weighted version of Proposition 3.1. For any smooth positively curved pair, fix $m \geq 1$ and a vector $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ independent of the initial point $z \in X$, and set $\xi_0 = \xi_{m+1} = 0$. Denote the corresponding second variation by $\mathcal{Q}_{m,\xi}(z)$. This is a valid fixed-endpoint variation: for each fixed finite physical segment, sufficiently small perturbations of its internal arclength parameters preserve all the strict physical phase inequalities. Empty sums below are understood to be zero.

The tridiagonal coefficients in (18)–(19) give

$$\mathcal{Q}_{m,\xi}(z) = \sum_{j=1}^m a_j(z) \xi_j^2 + 2 \sum_{j=1}^{m-1} b_j(z) \xi_j \xi_{j+1}.$$

Write d_{11} and d_{22} for the pure derivatives of the edge Lagrangian, according to its type. As in the proof of Proposition 3.1,

$$a_j(z) = d_{22}(\mathcal{T}^{j-1}z) + d_{11}(\mathcal{T}^jz), \quad b_j(z) = b(\mathcal{T}^jz).$$

These functions are bounded, and $\mu(X) < \infty$ by (22). Therefore all the following integrals exist. Measure invariance from Lemma 2.3 and the definitions in (23) imply

$$\int_X a_j \, d\mu = \mathcal{I} - 2\mathcal{B}, \quad \int_X b_j \, d\mu = \mathcal{B}.$$

Consequently,

$$(55) \quad \int_X \mathcal{Q}_{m,\xi} d\mu = (\mathcal{I} - 2\mathcal{B}) \sum_{j=1}^m \xi_j^2 + 2\mathcal{B} \sum_{j=1}^{m-1} \xi_j \xi_{j+1}.$$

This identity uses only finite sums and measure invariance; it requires neither ergodicity nor a sign assumption on the Hessians. Choosing $\xi_j = 1$ recovers (24).

Return to the balanced pair of the statement and put $\lambda = \sqrt{A_Q/A_P}$. Let X_λ and μ_λ denote the physical phase space and invariant measure for $(Q, \lambda P)$, and write $\mathcal{Q}_{m,\xi}^{(\lambda)}$ for its second variation in Euclidean arclength coordinates. Since

$$v_Q = 0, \quad A_{\lambda P} = A_Q, \quad \ell_Q \ell_{\lambda P} = 4\pi A_Q \Gamma,$$

(32) and (37) give

$$\mathcal{I}_\lambda = 4\pi A_Q (\Gamma - 1), \quad 2\mathcal{B}_\lambda = 4\pi A_Q \Gamma.$$

Thus (55) becomes

$$(56) \quad \int_{X_\lambda} \mathcal{Q}_{m,\xi}^{(\lambda)} d\mu_\lambda = 4\pi A_Q \left(- \sum_{j=1}^m \xi_j^2 + \Gamma \sum_{j=1}^{m-1} \xi_j \xi_{j+1} \right).$$

To optimize this expression, let C_m be the symmetric $m \times m$ matrix with zero diagonal and $1/2$ on each of its first off-diagonals; for $m = 1$, this means $C_1 = (0)$. Then

$$\xi^\top C_m \xi = \sum_{j=1}^{m-1} \xi_j \xi_{j+1}.$$

Put $\theta = \pi/(m+1)$. For $1 \leq k \leq m$, the nonzero vector $v^{(k)}$ with entries $v_j^{(k)} = \sin(kj\theta)$ satisfies

$$(C_m v^{(k)})_j = \frac{1}{2} (v_{j-1}^{(k)} + v_{j+1}^{(k)}) = \cos(k\theta) v_j^{(k)}, \quad 1 \leq j \leq m,$$

where $v_0^{(k)} = v_{m+1}^{(k)} = 0$. The m eigenvalues $\cos(k\theta)$ are distinct, so these vectors form an eigenbasis. The largest eigenvalue of C_m is therefore $\cos \theta$, attained by $v^{(1)}$. By (56),

$$(57) \quad \max_{\xi \in \mathbb{R}^m \setminus \{0\}} \frac{\int_{X_\lambda} \mathcal{Q}_{m,\xi}^{(\lambda)} d\mu_\lambda}{4\pi A_Q \sum_{j=1}^m \xi_j^2} = \Gamma \cos \theta - 1.$$

Take $\xi = v^{(1)}$, as in (54). The finite geometric sum $\sum_{j=0}^m e^{2ij\theta} = 0$ gives $\sum_{j=1}^m \cos(2j\theta) = -1$, and hence

$$\sum_{j=1}^m \xi_j^2 = \frac{m+1}{2}.$$

The eigenvector identity also gives

$$\sum_{j=1}^{m-1} \xi_j \xi_{j+1} = \cos \theta \sum_{j=1}^m \xi_j^2.$$

Substitution into (56) yields the exact formula

$$(58) \quad \int_{X_\lambda} \mathcal{Q}_{m,\xi}^{(\lambda)} d\mu_\lambda = 2\pi A_Q(m+1) \left(\Gamma \cos \frac{\pi}{m+1} - 1 \right) > 0.$$

The function $\mathcal{Q}_{m,\xi}^{(\lambda)}$ is continuous on X_λ , since it is formed from finitely many iterates of the smooth alternating map and derivatives of the edge Lagrangians. Its strictly positive integral implies that

$$U_\lambda = \{z \in X_\lambda : \mathcal{Q}_{m,\xi}^{(\lambda)}(z) > 0\}$$

is nonempty. This set is open and has positive physical measure, because the invariant density is strictly positive at every physical point. At each $z \in U_\lambda$, the nonzero vector (54) is a positive direction of the corresponding fixed-endpoint Hessian.

The orbit correspondence and invertible Hessian congruence in the proof of Lemma 4.2 hold without any sign assumption. They transfer this conclusion to (Q, P) , preserving the number of internal vertices and openness. The common linear change in (39), followed by the arclength reparametrizations, likewise transfers it to the original pair. Positive directions are preserved by these congruences, and each transported nonempty open set has positive physical measure.

Finally, since $\Gamma > 1$, one has $0 < \arccos(\Gamma^{-1}) < \pi/2$. The cosine is strictly decreasing on $[0, \pi]$, which proves the asserted equivalence of the two length conditions. $\square$

Remark 4.4 (Comparison and scope of the sine bound). The constant vector has Rayleigh quotient $(m-1)/m$ for C_m . Consequently,

$$\cos \frac{\pi}{m+1} \geq \frac{m-1}{m},$$

with strict inequality for $m \geq 3$, since then the constant vector is not an eigenvector of C_m . Thus (1) implies (53): the sine argument also proves the positive-direction conclusion of Theorem 1.2. The preceding constant-variation

proof gives the original bound, while the sine choice optimizes the averaged Rayleigh quotient.

The smallest integer satisfying the sine condition is

$$m_{\text{sine}}(\Gamma) = \left\lfloor \frac{\pi}{\arccos(\Gamma^{-1})} \right\rfloor.$$

In particular, as $\varepsilon \downarrow 0$,

$$m_{\text{sine}}(1 + \varepsilon) = \frac{\pi}{\sqrt{2\varepsilon}} + O(1),$$

because $\arccos((1 + \varepsilon)^{-1}) = \sqrt{2\varepsilon} + O(\varepsilon^{3/2})$. This is an inverse-square-root sufficient bound in $\Gamma - 1$, rather than the inverse-linear bound obtained from the constant variation.

Equation (57) establishes optimality only for positivity of the averaged quadratic form among weight vectors independent of the initial phase point, in the chosen balanced, equal-area normalization. It does not establish the smallest orbit length at which an actual Hessian has a positive direction. When (53) fails, this averaging argument alone does not exclude positive directions along individual segments. Both bounds count internal alternating vertices, so the total number of vertices is $m + 2$. Neither bound is asserted to be a sharp dynamical threshold or an estimate for entropy or a Lyapunov exponent.

5. Continuous invariant graphs and automatic curvature

The pair theorem has now been proved independently of integrability. We next supply the variational input for its Minkowski applications. Throughout this section, K is a compact convex body with smooth boundary and $0 \in \text{int } K$, and D is any compact smooth convex table with nonempty interior. Neither boundary is initially assumed to have positive curvature. We use q for the arclength parametrization of $Q = \partial D$.

5.1. *The non-grazing phase annulus and total integrability.* Let

$$q : \mathbb{R}/\ell_Q\mathbb{Z} \longrightarrow Q = \partial D$$

be a positive Euclidean arclength parametrization, and use its periodic lift to $\mathbb{R}$. Before strict convexity has been established, define

$$\Omega_D = \left\{ (s, u) : \begin{array}{l} s < u < s + \ell_Q, \\ (1 - \theta)q(s) + \theta q(u) \in \text{int } D \quad \text{for every } 0 < \theta < 1 \end{array} \right\}.$$

The non-grazing ordered-chord phase annulus is

$$\mathcal{C}_Q = \Omega_D / ((s, u) \sim (s + \ell_Q, u + \ell_Q)).$$

For a smooth convex boundary, a chord belongs to Ω_D if and only if

$$\omega(q'(s), q(u) - q(s)) > 0, \quad \omega(q(u) - q(s), q'(u)) > 0.$$

Indeed, a tangent line is a supporting line. If either inequality fails to be strict, the endpoints lie in one supporting face and the chord is contained in the boundary. Conversely, a chord meeting the interior is transverse to the supporting tangent at each endpoint. In particular, Ω_D is open.

We shall need that its horizontal and vertical sections are intervals. Let the supporting face at $q(s)$ have a lift parametrized by $[a_s, b_s]$, where

$$a_s \leq s \leq b_s < a_s + \ell_Q.$$

For a singleton face, $a_s = b_s = s$. The only boundary endpoints excluded from an interior chord starting at $q(s)$ are those on this supporting face. Consequently,

$$\{u : (s, u) \in \Omega_D\} = (b_s, a_s + \ell_Q).$$

Similarly,

$$\{s : (s, u) \in \Omega_D\} = (b_u - \ell_Q, a_u).$$

Thus both coordinate sections are intervals, without any positive-curvature assumption.

The correspondence

$$(s, u) \mapsto \left(q(s), \frac{q(u) - q(s)}{N_K(q(u) - q(s))} \right)$$

identifies $\mathcal{C}_Q$ with the phase space of inward-pointing Minkowski unit velocities based at Q . Its inverse takes an inward ray to its unique second boundary intersection. Existence and uniqueness follow from convexity and compactness, and the intersection is transverse because the ray passes through the interior. This correspondence preserves the base point $q(s)$, and therefore preserves continuous graphs over the boundary.

The generating function is

$$(59) \quad G(s, u) = N_K(q(u) - q(s)), \quad (s, u) \in \Omega_D.$$

The billiard map sends (s_{i-1}, s_i) to (s_i, s_{i+1}) , and its configurations satisfy

$$G_2(s_{i-1}, s_i) + G_1(s_i, s_{i+1}) = 0.$$

As in the twist-map formulation of the problem, “billiard map” refers to the single-valued invertible continuous dynamics on the entire non-grazing phase annulus. No differential nondegeneracy is used in the preliminary argument below. Multiplying a generating function by minus one leaves its Euler–Lagrange equation unchanged. We write $S = -G$, so that minimizing the S -action is equivalent to maximizing the G -action; all fixed-endpoint comparisons keep the same number of edges and fix the endpoints in the lifted boundary parameter.

Definition 5.1 (Total integrability). The billiard in D with gauge N_K is totally integrable if $\mathcal{C}_Q$ is foliated by continuous invariant essential graphs over the boundary coordinate. Each lifted leaf has the form

$$u = f(s), \quad f(s + \ell_Q) = f(s) + \ell_Q, \quad (s, f(s)) \in \Omega_D.$$

Invariance means that the billiard map maps each leaf onto itself.

Remark 5.2 (Phase coordinates). The endpoint/direction correspondence preserves the base point and therefore preserves graphs. In canonical coordinates for $S = -G$, the map is $(s, u) \mapsto (s, \pi)$ with $\pi = -S_1(s, u)$. When strict twist holds this is a fibre-preserving diffeomorphism, as computed in Section 6. The preliminary arguments here use only the endpoint annulus and the stipulated globally defined billiard homeomorphism, not strict differential twist.

The restriction of the billiard map to such a leaf is conjugate, under projection to the first coordinate, to the circle map induced by f . It is a circle homeomorphism, and the displayed periodicity makes its degree equal to one. Hence f is an increasing homeomorphism of $\mathbb{R}$.

Changing to Minkowski arclength does not change this definition. Indeed,

$$h(s) = \int_0^s N_K(q'(r)) dr$$

is a smooth strictly increasing change of parameter satisfying

$$h(s + \ell_Q) = h(s) + \int_0^{\ell_Q} N_K(q'(r)) dr.$$

The corresponding endpoint change $(s, u) \mapsto (h(s), h(u))$ conjugates the billiard maps and preserves invariant graphs and their full foliation.

After Lemma 5.5, the table D is strictly convex and

$$\Omega_D = \{(s, u) : s < u < s + \ell_Q\}.$$

Thus the phase annulus then becomes exactly the ordered-chord cylinder used in the alternating reduction.

5.2. Weak twist and continuous calibration.

LEMMA 5.3 (Smoothness of the gauge and weak twist). *Let $K \subset \mathbb{R}^2$ be a compact convex body containing the origin in its interior and having C^∞ boundary. Then N_K is C^∞ on $\mathbb{R}^2 \setminus \{0\}$. For every $v \neq 0$, its Hessian is positive semidefinite and annihilates v . Consequently, there is a number $\eta(v) \geq 0$ such that*

$$(60) \quad D^2 N_K(v)[\xi, \zeta] = \eta(v) \omega(v, \xi) \omega(v, \zeta) \quad (\xi, \zeta \in \mathbb{R}^2).$$

For any smooth convex table D , the generating function on Ω_D satisfies

$$G_{su}(s, u) \geq 0.$$

Proof. Every ray from the origin meets ∂K exactly once. If $n(x)$ is the outward unit normal at $x \in \partial K$, then

$$x \cdot n(x) > 0,$$

because the supporting line at x separates $\text{int } K$, and in particular a small disk about the origin, from its exterior. Thus radial directions are transverse to the boundary. Writing k for a regular boundary parametrization of K , the radial parametrization

$$(r, s) \mapsto rk(s), \quad r > 0,$$

is a smooth local diffeomorphism and is globally one-to-one modulo the period of k . Since $N_K(rk(s)) = r$, the gauge is smooth away from the origin.

Convexity implies that $D^2N_K(v)$ is positive semidefinite. Euler's identity for the positively one-homogeneous function N_K is

$$DN_K(v) \cdot v = N_K(v).$$

Differentiation gives

$$D^2N_K(v)[v, \xi] = 0 \quad \text{for every } \xi.$$

In dimension two, the space of covectors annihilating v is one-dimensional and is spanned by $\xi \mapsto \omega(v, \xi)$. This proves (60), with $\eta(v) \geq 0$. More explicitly,

$$\eta(v) = \frac{D^2N_K(v)[Jv, Jv]}{|v|^4},$$

so η is smooth away from the origin. For $r > 0$, the homogeneity formulas are

$$\begin{aligned} DN_K(rv) &= DN_K(v), & D^2N_K(rv) &= r^{-1}D^2N_K(v), \\ \eta(rv) &= r^{-3}\eta(v). \end{aligned}$$

For $v = q(u) - q(s)$, differentiation of the generating function gives

$$G_{su}(s, u) = -D^2N_K(v)[q'(s), q'(u)] = -\eta(v)\omega(v, q'(s))\omega(v, q'(u)).$$

On Ω_D , chord orientation gives

$$\omega(v, q'(s)) < 0, \quad \omega(v, q'(u)) > 0.$$

Therefore $G_{su} \geq 0$. □

LEMMA 5.4 (Calibration of a continuous invariant graph). *Let $\ell > 0$, and let*

$$\Omega \subset \{(x, y) : x < y < x + \ell\}$$

be an open set invariant under $(x, y) \mapsto (x + \ell, y + \ell)$, whose nonempty horizontal and vertical sections are intervals. Let $S \in C^2(\Omega)$ satisfy

$$S(x + \ell, y + \ell) = S(x, y), \quad S_{12} \leq 0.$$

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing homeomorphism satisfying

$$f(x + \ell) = f(x) + \ell, \quad (x, f(x)) \in \Omega,$$

and the invariance identity

$$S_2(f^{-1}(x), x) + S_1(x, f(x)) = 0.$$

Then there are constants $c, \alpha \in \mathbb{R}$ and an ℓ -periodic function $U \in C^1(\mathbb{R})$ such that

$$(61) \quad S(x, y) \geq \alpha - U(x) + U(y) + c(y - x) \quad ((x, y) \in \Omega),$$

with equality on the graph $y = f(x)$. If $S_{12} < 0$ throughout Ω , equality holds only on that graph.

Every finite graph configuration minimizes the S -action among admissible configurations with the same lifted endpoints and the same number of edges.

Proof. Put $g = f^{-1}$ and

$$a(x) = -S_1(x, f(x)).$$

The function a is continuous and ℓ -periodic. The invariance identity gives

$$a(y) = S_2(g(y), y).$$

Choose

$$c = \frac{1}{\ell} \int_0^\ell a(x) dx$$

and an ℓ -periodic C^1 function U satisfying

$$U'(x) = a(x) - c.$$

Define

$$\Phi(x, y) = S(x, y) + U(x) - U(y) - c(y - x).$$

Then

$$\partial_2 \Phi(x, y) = S_2(x, y) - S_2(g(y), y).$$

For fixed y , the points x and $g(y)$ belong to the same horizontal interval of Ω . The inequality $S_{12} \leq 0$ therefore implies that $\xi \mapsto S_2(\xi, y)$ is nonincreasing between them. Since f and g are increasing,

$$y < f(x) \implies g(y) < x, \quad y > f(x) \implies g(y) > x.$$

Consequently,

$$\partial_2 \Phi(x, y) \leq 0 \quad \text{for } y < f(x), \quad \partial_2 \Phi(x, y) \geq 0 \quad \text{for } y > f(x).$$

The vertical section of Ω at x is an interval. It follows that $f(x)$ is a global minimizer of $y \mapsto \Phi(x, y)$ on that section. When $S_{12} < 0$, the displayed derivative inequalities are strict away from $f(x)$, so the minimizer is unique.

It remains to prove that the minimum

$$\alpha(x) = \Phi(x, f(x))$$

is constant without differentiating f . Fix x . Since Ω is open and f is continuous, for all sufficiently small h both $(x+h, f(x))$ and $(x, f(x+h))$ belong to Ω . The minimizing property gives

$$\begin{aligned} \Phi(x+h, f(x+h)) - \Phi(x, f(x+h)) &\leq \alpha(x+h) - \alpha(x) \\ &\leq \Phi(x+h, f(x)) - \Phi(x, f(x)). \end{aligned}$$

Because Φ is C^1 and $f(x+h) \rightarrow f(x)$, both outer expressions equal

$$h \partial_1 \Phi(x, f(x)) + o(|h|).$$

This follows directly by integrating $\partial_1 \Phi$ along the short horizontal segments in a fixed compact rectangle contained in Ω . But

$$\partial_1 \Phi(x, f(x)) = S_1(x, f(x)) + U'(x) + c = -a(x) + a(x) - c + c = 0.$$

Hence α is differentiable at every x , with $\alpha'(x) = 0$. It is therefore constant. This proves (61).

Finally, summing that inequality along an admissible configuration $y_m, \dots, y_n$ yields

$$\sum_{i=m}^{n-1} S(y_i, y_{i+1}) \geq (n-m)\alpha - U(y_m) + U(y_n) + c(y_n - y_m).$$

Equality holds for a graph configuration. The right-hand side depends only on the lifted endpoints and the number of edges, proving the minimizing property. $\square$

5.3. Positivity of table curvature.

LEMMA 5.5 (Total integrability excludes zero curvature). *Let D be a smooth convex table and let N_K be a smooth convex gauge as above. If the billiard in D is totally integrable, then the Euclidean curvature of ∂D is strictly positive at every point.*

Proof. By Lemma 5.3, $S = -G$ satisfies $S_{12} \leq 0$ on Ω_D . The coordinate sections of Ω_D are intervals. Each invariant graph has an increasing homeomorphism lift, as established in Subsection 5.1. Lemma 5.4 therefore applies to every leaf.

Since the foliation covers the whole phase annulus, every finite billiard orbit segment maximizes the fixed-endpoint G -action among admissible configurations with the same number of edges. In particular, every one-vertex second variation is nonpositive.

Suppose, for a contradiction, that the curvature vanishes at $x = q(s_0)$. Write

$$T = q'(s_0).$$

The parametrization is by Euclidean arclength, so

$$q''(s_0) = 0.$$

Let v be any vector in the open inward half-plane

$$\mathcal{H} = \{v : \omega(T, v) > 0\}.$$

The ray $x + \rho v$, $\rho > 0$, enters D and has a unique second boundary intersection $q(s_+)$, with

$$q(s_+) - x = \rho v$$

for some $\rho > 0$. The state (s_0, s_+) belongs to one leaf of the full foliation. Let s_- be its predecessor on that leaf. Thus s_-, s_0, s_+ is a billiard orbit segment.

Set

$$v_- = x - q(s_-), \quad v_+ = q(s_+) - x.$$

Keeping s_- and s_+ fixed, consider

$$A(r) = N_K(q(r) - q(s_-)) + N_K(q(s_+) - q(r)).$$

Both adjacent chords remain admissible for r close to s_0 , since Ω_D is open. The maximizing property gives $A''(s_0) \leq 0$. Direct differentiation gives

$$\begin{aligned} A''(s_0) &= D^2 N_K(v_-)[T, T] + D^2 N_K(v_+)[T, T] \\ &\quad + (DN_K(v_-) - DN_K(v_+)) \cdot q''(s_0) \\ &= D^2 N_K(v_-)[T, T] + D^2 N_K(v_+)[T, T]. \end{aligned}$$

Both terms are nonnegative by convexity. They must therefore both vanish. Positive homogeneity gives

$$D^2 N_K(\rho v) = \rho^{-1} D^2 N_K(v),$$

and hence

$$D^2 N_K(v)[T, T] = 0 \quad \text{for every } v \in \mathcal{H}.$$

For $v \in \mathcal{H}$, the vectors v and T are linearly independent. By (60),

$$0 = D^2 N_K(v)[T, T] = \eta(v) \omega(v, T)^2$$

forces $\eta(v) = 0$. Thus

$$D^2 N_K(v) = 0 \quad (v \in \mathcal{H}).$$

Since $\mathcal{H}$ is connected and convex, N_K is affine there:

$$N_K(v) = a \cdot v + b.$$

Positive one-homogeneity forces $b = 0$. Continuity at the two nonzero boundary vectors T and $-T$ now gives

$$N_K(T) = a \cdot T, \quad N_K(-T) = -a \cdot T.$$

Consequently,

$$N_K(T) + N_K(-T) = 0,$$

contradicting positivity of a Minkowski gauge on nonzero vectors. No zero-curvature point can exist. $\square$

Remark 5.6. This conclusion concerns the table D . The argument does not require $D = K$, and it does not assert positive curvature of an independent metric ball. When $D = K$, however, it supplies exactly the metric regularity needed for the polar reduction.

5.4. *Strict second variation for twist graphs.* Once strict twist is available, the same calibration also yields strict second variation. We record the consequence for later use.

Let $\ell > 0$ and

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x < y < x + \ell\}.$$

Let $S \in C^2(\mathcal{D})$ satisfy diagonal periodicity

$$S(x + \ell, y + \ell) = S(x, y)$$

and the negative twist condition

$$(62) \quad S_{12}(x, y) < 0.$$

The associated twist map on ordered pairs is defined by

$$(63) \quad S_2(x_{i-1}, x_i) + S_1(x_i, x_{i+1}) = 0.$$

An invariant essential graph has a lift

$$(64) \quad y = f(x), \quad f(x + \ell) = f(x) + \ell, \quad x < f(x) < x + \ell.$$

The restriction of the twist map to the graph is conjugate, under projection to the first coordinate, to the circle map induced by f . It is a degree-one circle homeomorphism, so f is an increasing homeomorphism of $\mathbb{R}$. No differentiability or Lipschitz regularity of f is needed: the continuous calibration lemma proved above applies directly.

PROPOSITION 5.7 (Calibration by an invariant graph). *Let S satisfy (62), and let (64) be an invariant essential graph. There exist a constant $c \in \mathbb{R}$, an ℓ -periodic C^1 function u , and a constant $\alpha \in \mathbb{R}$ such that*

$$(65) \quad S(x, y) \geq \alpha - u(x) + u(y) + c(y - x)$$

for every $(x, y) \in \mathcal{D}$, with equality if and only if $y = f(x)$. Consequently, every finite orbit segment on the graph minimizes the S -action among all sufficiently close finite configurations with the same endpoints; in fact it minimizes among every admissible configuration with the same lifted endpoints and the same number of edges.

Proof. Apply Lemma 5.4 with

$$\Omega = \mathcal{D} = \{(x, y) : x < y < x + \ell\}.$$

The invariance identity in that lemma is exactly the Euler–Lagrange equation restricted to the graph. Since $S_{12} < 0$, equality in the calibration inequality holds if and only if $y = f(x)$. The fixed-endpoint minimizing property follows by summing the inequality along configurations with the same number of edges. $\square$

The calibration gives semidefiniteness. The following elementary extension argument gives strict definiteness without importing a separate nonconjugacy theorem.

All fixed-endpoint Hessians are taken in the internal vertex parameters. Admissibility imposes no additional infinitesimal restriction, because each finite physical configuration belongs to an open set of admissible configurations. When there are no internal vertices, definiteness on the zero-dimensional variation space is understood vacuously.

LEMMA 5.8 (A calibrated twist segment has strict second variation). *Under the hypotheses of Proposition 5.7, the fixed-endpoint Hessian of the S -action along every finite graph orbit segment is positive definite. Equivalently, for $G = -S$ it is negative definite.*

Proof. Calibration gives positive-semidefinite Hessians on every finite segment of the bi-infinite graph orbit. The off-diagonal coefficients are $S_{12} \neq 0$. Apply the positive version of Lemma 2.2. $\square$

6. Polar reduction and variational transfer

In this section K is a smooth positively curved unit ball and D is a smooth positively curved table. The table curvature is supplied by Lemma 5.5 whenever total integrability is assumed. We shall apply the pair theorem to

$$Q = \partial D, \quad P = \partial K^\omega.$$

6.1. *Support representation and regularity.* For a convex body K containing the origin in its interior, its ordinary polar is

$$K^\circ = \{u \in \mathbb{R}^2 : \langle u, x \rangle \leq 1 \text{ for all } x \in K\},$$

and its symplectic polar is

$$(66) \quad K^\omega = \{p \in \mathbb{R}^2 : \omega(p, x) \leq 1 \text{ for all } x \in K\}.$$

By (7),

$$(67) \quad K^\omega = -JK^\circ.$$

Thus symplectic polarity differs from ordinary polarity only by a quarter-turn. If ∂K is smooth and strongly convex, then so is ∂K^ω .

LEMMA 6.1 (Gauge as a symplectic support function). *Let K be a compact convex body with smooth boundary and 0 in its interior. For every $v \in \mathbb{R}^2$,*

$$(68) \quad N_K(v) = \max_{p \in K^\omega} \omega(p, v).$$

For $v \neq 0$, the maximizer belongs to ∂K^ω and is unique.

Proof. For $v = 0$, both sides vanish. We therefore assume $v \neq 0$ in the remainder of the proof. The linear isomorphism

$$p \longmapsto [x \longmapsto \omega(p, x)]$$

identifies K^ω with the polar body of K in the dual vector space. If $v \in rK$, then every $p \in K^\omega$ satisfies $\omega(p, v) \leq r$; hence the right-hand side of (68) is at most $N_K(v)$. Conversely, put $r = N_K(v)$ and $x = v/r \in \partial K$. A supporting covector to K at x , normalized to take the value 1 at x , corresponds to some $p \in K^\omega$ with $\omega(p, v) = r$. This proves equality. Smoothness of ∂K makes K^ω strictly convex, so the maximizer is unique for $v \neq 0$. $\square$

LEMMA 6.2 (Equivariance of symplectic polarity). *For every $A \in \text{SL}(2, \mathbb{R})$ and every convex body K containing the origin,*

$$(69) \quad (AK)^\omega = A(K^\omega).$$

Proof. By (39),

$$\omega(p, Ax) = \omega(A^{-1}p, x).$$

Thus $p \in (AK)^\omega$ if and only if $\omega(A^{-1}p, x) \leq 1$ for every $x \in K$, which is equivalent to $A^{-1}p \in K^\omega$. $\square$

LEMMA 6.3 (Strict twist and polar regularity). *Suppose that K and D have smooth boundaries of positive curvature. Then*

$$\Omega_D = \{(s, u) : s < u < s + \ell_Q\}, \quad G_{su} > 0,$$

and ∂K^ω is smooth and positively curved.

Proof. Positive curvature of D makes it strictly convex, so every ordered pair of distinct boundary points determines an interior chord. This gives the asserted description of Ω_D .

Let $k(s)$ be a Euclidean arclength parametrization of ∂K . Differentiate $N_K(k(s)) = 1$ twice. Since $k'' = \kappa_K Jk'$ and $DN_K(k)$ is an outward normal,

$$D^2 N_K(k(s))[k'(s), k'(s)] = \kappa_K(s) |DN_K(k(s))| > 0.$$

The radial vector $k(s)$ is not tangent to the boundary, so (60) implies $\eta(k(s)) > 0$. Every nonzero vector is a positive multiple of a boundary point. Homogeneity thus gives $\eta(v) > 0$ for $v \neq 0$. Applying the mixed-derivative formula in Lemma 5.3 to chords of D gives $G_{su} > 0$.

We also verify the polar regularity explicitly. Identify covectors with Euclidean vectors and put $z(s) = DN_K(k(s))$. Convexity and Euler's identity give

$$z(s) \cdot y \leq 1 \quad (y \in K), \quad z(s) \cdot k(s) = 1.$$

Thus $z(s) \in \partial K^\circ$. Conversely, every point of ∂K° is a normalized supporting covector of K , and therefore equals $DN_K(k(s))$ at its unique contact point. Hence z parametrizes the whole polar boundary.

Equation (60) gives

$$z'(s) = \sigma(s) Jk(s), \quad \sigma(s) = \eta(k(s)) \omega(k(s), k'(s)) > 0.$$

Here $\omega(k, k') > 0$ follows from $0 \in \text{int } K$ and the supporting-line property. In particular $z' \neq 0$, and

$$\omega(z'(s), z''(s)) = \sigma(s)^2 \omega(k(s), k'(s)) > 0.$$

The polar boundary is smooth and has positive curvature. The same holds for ∂K^ω , since $K^\omega = -JK^\circ$. $\square$

For a strongly convex metric and table, the endpoint and canonical graph formulations can also be identified smoothly. For $S = -G$, set

$$\pi = -S_1(s, u) = G_1(s, u).$$

Since $\partial_u \pi = G_{12} > 0$, the map

$$(s, u) \mapsto (s, \pi)$$

is a fibre-preserving diffeomorphism onto its image. Its image is the canonical annulus

$$\{(s, \pi) : -N_K(q'(s)) < \pi < N_K(-q'(s))\}.$$

The endpoint limits follow from $G_1(s, u) = -DN_K(q(u) - q(s)) \cdot q'(s)$, positive homogeneity, and Euler's identity. The Euler–Lagrange equation identifies this coordinate change with the usual generating-function description

$$\pi = -S_1(s, u), \quad \Pi = S_2(s, u).$$

It therefore conjugates the billiard maps and preserves continuous invariant graphs and their full foliation.

6.2. *Exact elimination of the polar vertices.* For fixed s, u , define

$$(70) \quad H(s, t, u) = L_0(s, t) + L_1(t, u) = \omega(p(t), q(u) - q(s)).$$

By Lemma 6.1,

$$(71) \quad \max_t H(s, t, u) = N_K(q(u) - q(s)) = G(s, u).$$

The maximizing parameter $t = t(s, u)$ is unique. At it,

$$(72) \quad q(u) - q(s) = \rho p'(t) \quad \text{for a unique } \rho > 0.$$

Indeed, $H_t = 0$ makes the chord parallel to $p'(t)$; the positive sign is forced by maximality, since by (8)

$$(73) \quad H_{tt} = \omega(p''(t), q(u) - q(s)) = \rho \kappa_P(t) \omega(Jp'(t), p'(t)) = -\rho \kappa_P(t) < 0.$$

Here both curves are arclength parametrized. The implicit-function theorem therefore makes $t(s, u)$ smooth.

The map

$$(74) \quad \Psi : \mathcal{C}_Q \longrightarrow \mathcal{P}_0, \quad \Psi(s, u) = (s, t(s, u)),$$

is a diffeomorphism. Its inverse takes (s, t) to the unique second intersection u in (14). This identifies every ordered Q -chord with its unique supporting point on P .

This correspondence covers the entire physical component, not merely a neighborhood of the diagonal. Indeed, for every $(s, t) \in \mathcal{P}_0$, the ray with direction $p'(t)$ enters the table D and has a unique second intersection $q(u)$. At that intersection,

$$\omega(p'(t), q'(u)) > 0,$$

so the intersection is transverse and depends smoothly on (s, t) . Conversely, every ordered Q -chord has its unique support maximizer on P .

The alternating maps are invertible on their whole physical components. For example, given $(t, u) \in \mathcal{P}_1$, the ray

$$q(u) - \rho p'(t), \quad \rho > 0,$$

is inward-pointing at $q(u)$ and recovers the unique preceding Q -vertex. The other inverse is obtained in the same way. Consequently, every bi-infinite physical alternating orbit projects to a bi-infinite Minkowski orbit, and every Minkowski orbit has its unique alternating lift.

PROPOSITION 6.4 (Exact alternating reduction). *Under the phase diffeomorphism (74), the two-step alternating map $T_1 \circ T_0$ is exactly the Minkowski billiard map in D generated by G in (59). Moreover,*

$$(75) \quad G_{su}(s, u) > 0 \quad \text{on } \mathcal{C}_Q.$$

Proof. For an alternating finite action with Q -vertices s_i and intervening P -vertices t_i , the variables t_i occur independently. Maximizing each of them and using (71) gives

$$\sum_i [L_0(s_i, t_i) + L_1(t_i, s_{i+1})] \mapsto \sum_i G(s_i, s_{i+1}).$$

At the unique maximizing branch, the envelope theorem identifies the derivatives of the reduced action with those of the full action. Hence the reduced Euler–Lagrange equation is equivalent to the pair of alternating Euler–Lagrange equations, and the two-step map is exactly the Minkowski map.

For the twist sign, use $H_t(s, t(s, u), u) = 0$ and differentiate. Since $H_{su} = 0$, the second-order envelope formula gives

$$(76) \quad G_{su} = -\frac{H_{st}H_{tu}}{H_{tt}}.$$

At the physical maximizing point,

$$H_{st} = \omega(q'(s), p'(t)) > 0, \quad H_{tu} = \omega(p'(t), q'(u)) > 0$$

by (12), (72), and Lemma 2.1. Together with (73), this proves (75). $\square$

Remark 6.5. No self condition has entered this reduction. Its two curves are ∂D and ∂K^ω . It uses the same symplectic support representation underlying the correspondence in [3]; the explicit construction also identifies the physical component and the sign of its Hessian.

COROLLARY 6.6 (Strict maximality for the reduced action). *For the strongly convex metric and table of this section, if the Minkowski billiard is totally integrable, then every finite Minkowski orbit segment is a strict local maximum of the fixed-endpoint G -action. Its fixed-endpoint Hessian is negative definite.*

Proof. By Proposition 6.4, $G_{12} > 0$, so $S = -G$ satisfies (62). Every orbit lies on one leaf of the full invariant-graph foliation. Proposition 5.7 and Lemma 5.8 apply to S , and changing sign gives the conclusion for G . $\square$

6.3. Schur-complement transfer. Consider a physical alternating orbit segment

$$(77) \quad s_0, t_0, s_1, t_1, \dots, t_{m-1}, s_m,$$

whose endpoints s_0, s_m lie on Q . Its alternating action is

$$(78) \quad \mathcal{A}(s_0, t_0, s_1, \dots, t_{m-1}, s_m) = \sum_{i=0}^{m-1} [L_0(s_i, t_i) + L_1(t_i, s_{i+1})].$$

Eliminating the unique support maximizers $t_i = t(s_i, s_{i+1})$ gives the reduced action

$$(79) \quad \mathcal{G}(s_0, \dots, s_m) = \sum_{i=0}^{m-1} G(s_i, s_{i+1}).$$

All Hessians below are taken in scalar curve parameters at the critical configuration.

LEMMA 6.7 (Schur-complement identity). *Fix s_0, s_m . Order the internal variables in (78) as*

$$(s_1, \dots, s_{m-1}; t_0, \dots, t_{m-1}).$$

At a physical critical segment write

$$(80) \quad \text{Hess } \mathcal{A} = \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}.$$

Then D is diagonal and negative definite, and

$$(81) \quad \text{Hess } \mathcal{G} = A - BD^{-1}B^\top.$$

Consequently, $\text{Hess } \mathcal{A}$ is negative semidefinite, respectively negative definite, if and only if $\text{Hess } \mathcal{G}$ has the corresponding property.

Proof. Different t_i variables do not interact, so D is diagonal. Its i th diagonal entry is

$$\frac{\partial^2}{\partial t_i^2} [L_0(s_i, t_i) + L_1(t_i, s_{i+1})] = H_{tt}(s_i, t_i, s_{i+1}),$$

which is strictly negative by (73). The critical equations in the t -variables can therefore be solved smoothly for $t = t(s)$.

Differentiating those equations gives

$$D \frac{\partial t}{\partial s} + B^\top = 0, \quad \frac{\partial t}{\partial s} = -D^{-1}B^\top.$$

A second differentiation of the reduced action yields (81). Equivalently, this is the standard second-order envelope formula.

Finally, the congruence

$$(82) \quad \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix} \begin{pmatrix} I & 0 \\ -D^{-1}B^\top & I \end{pmatrix} = \begin{pmatrix} A - BD^{-1}B^\top & 0 \\ 0 & D \end{pmatrix}$$

shows that the inertia of the full Hessian is the sum of the inertias of the two diagonal blocks. Since D is negative definite, the asserted equivalences follow. $\square$

PROPOSITION 6.8 (Lift of negative-definite second variation). *Let D and K be as in this section. If the Minkowski billiard in D with gauge N_K is totally integrable, then the fixed-endpoint Hessian of the alternating action is negative definite along every finite physical orbit segment of $(\partial D, \partial K^\omega)$. In particular, every such segment is a strict local maximum of the fixed-endpoint action.*

Proof. By the global phase correspondence in Proposition 6.4, every physical alternating orbit projects to an orbit of the Minkowski map in D . Since the invariant-graph foliation fills its whole phase annulus, the reduced second-variation conclusion applies to every such projected orbit.

For a segment with both endpoints on Q , Corollary 6.6 makes the reduced Hessian in (79) negative definite. Lemma 6.7 then makes the full alternating Hessian negative definite.

A segment with any other choice of endpoint parities can be extended along the same orbit by at most one vertex at either end until both enlarged endpoints lie on Q . The original fixed-endpoint Hessian is a principal submatrix of the enlarged negative-definite Hessian. Principal submatrices of a negative-definite matrix are negative definite, proving the assertion. $\square$

Remark 6.9 (Jacobi fields). For a scalar discrete Lagrangian, the kernel of a fixed-endpoint Hessian is exactly the space of Jacobi fields vanishing at the two endpoints. Thus Proposition 6.8 implies the usual absence of conjugate points. We retain the sign-sensitive formulation by fixed-endpoint Hessians. The finite-block argument of Section 3 needs only their nonpositivity; Appendix A uses strict definiteness to construct the Green field. This is the alternating analogue of the maximizing convention in [8] and [7]; changing the sign of the action gives the minimizing convention of [11].

7. Classification of metric–table pairs

The polar reduction now applies the general pair theorem to a billiard whose table and gauge are independent. Their remaining freedom is a common Euclidean normal form together with an exact linear term in the action.

Proof of Theorem 1.3. Assume total integrability. Lemma 5.5 gives positive curvature of ∂D . The unit ball K is strongly convex by the present hypothesis, so $P = \partial K^\omega$ is smooth and positively curved by the polar calculation in Lemma 6.3. Apply Proposition 6.4 with $Q = \partial D$. Continuous calibration and the extension lemma give negative-definite reduced Hessians; the Schur-complement argument then gives the same property for every alternating segment. Theorem 1.1 yields

$$AD = c_0 + rB_2, \quad A(K^\omega) = d + \rho B_2$$

for some $A \in \text{SL}(2, \mathbb{R})$ and $r, \rho > 0$. Since 0 is in the interior of K^ω , $|d| < \rho$. Symplectic invariance and the support formula give

$$\begin{aligned} N_K(x) &= \max_{p \in K^\omega} \omega(p, x) \\ &= \max_{p' \in d + \rho B_2} \omega(p', Ax) = \omega(d, Ax) + \rho |Ax|. \end{aligned}$$

Set $B = \rho A$, $\beta = Jd/\rho$, $c = \rho c_0$, and $R = \rho r$. Then $|\beta| < 1$ and (3) follows.

Conversely, in the coordinates $y = Bx$, the edge action is

$$|y_1 - y_0| + \beta \cdot (y_1 - y_0).$$

The second term is the exact endpoint difference $\beta \cdot y_1 - \beta \cdot y_0$. It does not alter the Euler–Lagrange equation or any fixed-endpoint Hessian. The table is the round disk $c + RB_2$. In angular coordinates its Euclidean edge action is

$$G_{\text{disk}}(\theta, \phi) = 2R \sin \frac{\phi - \theta}{2}, \quad 0 < \phi - \theta < 2\pi.$$

The Euler–Lagrange equation is

$$\cos \frac{\theta_i - \theta_{i-1}}{2} = \cos \frac{\theta_{i+1} - \theta_i}{2}.$$

Both half-increments belong to $(0, \pi)$, where cosine is strictly decreasing. Thus consecutive angular increments are equal. The invariant graphs $\phi = \theta + \delta$, $0 < \delta < 2\pi$, foliate the whole non-grazing annulus. The exact linear term leaves this dynamics unchanged, and the linear change of table coordinates preserves the graph condition. This proves the converse. $\square$

7.1. The exact linear freedom. The condition $|\beta| < 1$ in (3) guarantees that the gauge is positive on nonzero vectors. In the normalized coordinates,

$$(1 - |\beta|)|y| \leq |y| + \beta \cdot y \leq (1 + |\beta|)|y|.$$

Its linear part contributes

$$\sum_{i=0}^{m-1} \beta \cdot (y_{i+1} - y_i) = \beta \cdot y_m - \beta \cdot y_0$$

to every finite action. Thus it is invisible to both the reflection law and fixed-endpoint second variation. This elementary freedom is part of the more general equivalence problem studied in [1].

If the metric is reversible, then $N_K(x) = N_K(-x)$ in (3) forces $\beta = 0$. The unit ball is then $B^{-1}B_2$, and the table is a translate and positive dilation of it. Without reversibility, $\beta \neq 0$ gives actual totally integrable metric–table pairs. Theorem 1.3 therefore classifies a larger family than the self-Minkowski corollary. It is precisely the constraint $D = K$ that rules out these nonzero linear terms.

8. Self-Minkowski rigidity

We now impose $D = K$. Automatic table curvature becomes automatic metric curvature, so Theorem 1.3 applies without a positive-curvature hypothesis in the original statement. The remaining geometric fact fixes the center.

LEMMA 8.1 (Polar of a translated disk). *Let*

$$D(c, r) = \{x \in \mathbb{R}^2 : |x - c| \leq r\}, \quad |c| < r.$$

If the boundary of $D(c, r)^\omega$ is a Euclidean circle, then $c = 0$.

Proof. Since symplectic polarity differs from ordinary polarity by a Euclidean rotation, it suffices to consider $D(c, r)^\circ$. Its boundary is

$$c \cdot u + r|u| = 1.$$

Squaring gives

$$u^\top(r^2 I - cc^\top)u + 2c \cdot u - 1 = 0.$$

Conversely, every solution of this quadratic equation satisfies $1 - c \cdot u > 0$. Indeed, otherwise

$$r|u| = c \cdot u - 1 < c \cdot u \leq |c| |u| < r|u|,$$

a contradiction. Thus squaring has introduced no additional branch.

Set

$$M = r^2 I - cc^\top.$$

The assumption $|c| < r$ makes M positive definite. Completing the square gives

$$(u + M^{-1}c)^\top M(u + M^{-1}c) = 1 + c^\top M^{-1}c.$$

The right-hand side is strictly positive. The squared semiaxis lengths of this ellipse are that positive number divided by the eigenvalues of M . Those eigenvalues are

$$r^2, \quad r^2 - |c|^2.$$

The ellipse is a Euclidean circle only when they agree. Hence $|c| = 0$. $\square$

Proof of Corollary 1.4. Suppose that the self-Minkowski billiard is totally integrable. Lemma 5.5, applied with $D = K$, shows that ∂K has positive curvature everywhere. We may therefore apply Theorem 1.3. It gives an orientation-preserving linear isomorphism B , a vector β with $|\beta| < 1$, and a disk $D(c, R)$ such that, in the coordinates $y = Bx$,

$$K' = BK = D(c, R), \quad N_{K'}(y) = |y| + \beta \cdot y.$$

By the support representation of Lemma 6.1,

$$N_{K'}(y) = \max_{p \in -J\beta + B_2} \omega(p, y),$$

since $\omega(-J\beta, y) = \beta \cdot y$. A compact convex body is determined by its support function, so

$$(K')^\omega = -J\beta + B_2.$$

For completeness, uniqueness follows from separation by a line: a point outside a compact convex body is detected by a linear functional, and every linear functional on the plane has the form $\omega(\cdot, y)$ for some y . Thus the polar of $D(c, R)$ is a disk. Lemma 8.1 gives $c = 0$.

The gauge of RB_2 is $|y|/R$, hence

$$|y| + \beta \cdot y = \frac{|y|}{R} \quad (y \in \mathbb{R}^2).$$

Adding the equations for y and $-y$ gives $R = 1$; subtracting them gives $\beta = 0$. Therefore

$$N_K(x) = |Bx| = \sqrt{x^\top B^\top Bx},$$

which is (4) with $H = B^\top B$ positive definite.

Conversely, if $N_K(x) = \sqrt{x^\top Hx}$, let $B = H^{1/2}$. Then $BK = B_2$ and the data in Theorem 1.3 have $\beta = 0$, $c = 0$, and $R = 1$. Its converse, with the full constant-separation foliation of the round disk, proves total integrability. $\square$

8.1. *The formulation in Question 2.* The billiard in [2, pp. 1–2] is generated by $N(\gamma(s_1) - \gamma(s_0))$ and is totally integrable when its whole phase space is foliated by invariant graphs. With $\gamma = \partial K$, this is (59) and Definition 5.1. The change from N -arclength to Euclidean arclength in Section 5.1 preserves the graph condition, as does the endpoint/direction correspondence. Smoothness of the gauge away from 0 follows from radial transversality, proved in Lemma 5.3; it is not a hidden strong-convexity assumption.

The starting dynamics is the single-valued invertible billiard map on its entire non-grazing annulus, as in the question. No reflection branch is added in place of that map. Before positive curvature is proved, chords contained in a supporting face are excluded as grazing, and the admissible endpoint sections remain intervals. This is the only domain property needed by the continuous calibration.

Accordingly, Corollary 1.4 requires neither reversibility nor a finite symmetry group, and imposes no analyticity, perturbative smallness, or positive-curvature hypothesis. It does not replace total integrability by a first integral on a proper subannulus or an open dense subset. In its conclusion, “Euclidean” means induced by an inner product, not necessarily the coordinate norm $|x|$.
 AI use statement. ChatGPT (including GPT-5.5, GPT-5.6, and GPT-6) was used to convert handwritten notes into L^AT_EX, assist with literature searches, critically assess proposed approaches, help develop and refine mathematical arguments, and review the manuscript.

In particular, during the development of the proof two internal bottlenecks were referred to as *Gap R* and *Gap N*. Gap R concerned the reversible-case rigidity step. ChatGPT contributed to the development of the route that ultimately resolved this bottleneck, in particular the finite-block averaging mechanism of Section 3 and the two-curve affine-rigidity argument of Section 4. These arguments replaced earlier proposed routes based on direct Fourier-positivity or related local defects.

Gap N concerned the nonreversible odd part of the gauge. ChatGPT contributed to the argument identifying the remaining nonreversible freedom with an exact linear endpoint coboundary in Section 7, and to the self-polarity argument eliminating this freedom in Section 8. The explicit computation of the odd part of the gauge of a translated disk is recorded in Appendix B.

ChatGPT also contributed to the development and refinement of the continuous-invariant-graph calibration and automatic-curvature argument in Section 5, including Lemmas 5.4 and 5.5, and to the polar reduction and variational transfer in Section 6, including the Schur-complement transfer in Section 6.3. During the research process, ChatGPT was also used to test and reject several proposed approaches, including a midpoint-defect strategy; these discarded approaches are not used in the manuscript.

All mathematical statements and proofs used in the final manuscript, and all citations and references, were independently checked by the author. The author takes full responsibility for the correctness and final content of the work, including material developed with the assistance of generative AI.

Appendix A. An independent Green–Riccati derivation

This appendix is not used in the finite-block proof. It records a second derivation of (25), in the discrete Hopf framework of [8], [11], and [7], with all limits and integrability bounds explicit.

Assume that every finite physical alternating segment has a negative-definite fixed-endpoint Hessian. Retain the notation of Sections 2 and 3. In particular, the two-edge segment with x_n as its sole internal vertex gives

$$(83) \quad a_n < 0 \quad \text{for every } n.$$

A scalar Jacobi field satisfies

$$(84) \quad b_{n-1}\xi_{n-1} + a_n\xi_n + b_n\xi_{n+1} = 0.$$

A.1. *The Green–Riccati limit.* Fix an integer m . Let $\xi^{(m)}$ be the Jacobi solution initialized by

$$(85) \quad \xi_m^{(m)} = 0, \quad \xi_{m+1}^{(m)} = 1,$$

and define, whenever $k \geq m + 1$,

$$(86) \quad R_k^{(m)} = b_k \frac{\xi_{k+1}^{(m)}}{\xi_k^{(m)}}.$$

LEMMA A.1 (Monotone Riccati limit). *Under strict second-variation maximality of every finite alternating segment, the quantities in (86) are well defined and positive. They satisfy*

$$(87) \quad R_{m+1}^{(m)} = -a_{m+1}, \quad R_k^{(m)} = -a_k - \frac{b_{k-1}^2}{R_{k-1}^{(m)}} \quad (k \geq m + 2).$$

For each fixed k , $R_k^{(m)}$ decreases as $m \rightarrow -\infty$ and converges to a limit

$$(88) \quad R_k = \lim_{m \rightarrow -\infty} R_k^{(m)} \in (0, \infty).$$

The limits satisfy the Riccati equation

$$(89) \quad R_k = -a_k - \frac{b_{k-1}^2}{R_{k-1}}.$$

They depend Borel measurably on the phase point and equivariantly under the alternating map.

Proof. The recurrence (87) follows directly from (84). To prove positivity, consider the negative of the fixed-endpoint Hessian on the internal vertices $m + 1, \dots, k$. It is positive definite. Its successive $LDL^\top$ pivots are precisely

$$R_{m+1}^{(m)}, R_{m+2}^{(m)}, \dots, R_k^{(m)},$$

so all are positive. Equivalently, positivity follows inductively from the Schur complements of the leading principal blocks. It also follows that every $\xi_j^{(m)}$ with $j > m$ is positive.

Move the left endpoint from m to $m - 1$. Then

$$R_{m+1}^{(m-1)} = -a_{m+1} - \frac{b_m^2}{-a_m} < -a_{m+1} = R_{m+1}^{(m)}.$$

Each map

$$r \mapsto -a_k - \frac{b_{k-1}^2}{r}$$

is strictly increasing on $(0, \infty)$, so induction gives $R_k^{(m-1)} < R_k^{(m)}$ for every fixed $k > m + 1$. Thus the net decreases as the left endpoint tends to $-\infty$.

The limit cannot vanish. Positivity of the next pivot and (87) give

$$0 < R_{k+1}^{(m)} = -a_{k+1} - \frac{b_k^2}{R_k^{(m)}},$$

hence

$$(90) \quad R_k^{(m)} > \frac{b_k^2}{-a_{k+1}} > 0.$$

For every $m \leq k-1$,

$$0 < R_k^{(m)} \leq -a_k,$$

with equality in the upper bound when $m = k-1$, and strict inequality when $m \leq k-2$. Together with the preceding lower bound, this gives

$$\frac{b_k^2}{-a_{k+1}} \leq R_k \leq -a_k.$$

The left-hand side is strictly positive at the fixed phase point. Thus R_k is finite and nonzero. Passing to the limit in the recurrence is legitimate because $R_{k-1} > 0$, and gives

$$R_k = -a_k - \frac{b_{k-1}^2}{R_{k-1}}.$$

In particular, the upper bound for the limiting field is strict.

Let $\mathcal{T}$ denote the one-step alternating map on $\mathcal{P}_0 \sqcup \mathcal{P}_1$. For each integer $M \geq 1$, the finite-past quantity $R_0^{(-M)}$ is a continuous function of the phase point: it is formed from finitely many iterates of the smooth diffeomorphism $\mathcal{T}$, using divisions by strictly positive finite pivots. Therefore

$$R(z) = \lim_{M \rightarrow \infty} R_0^{(-M)}(z)$$

is Borel measurable. Shifting the orbit shifts the finite-past construction; letting its left endpoint tend to minus infinity gives

$$R_k(z) = R(\mathcal{T}^k z).$$

This proves equivariance. No lower bound uniform over the whole open phase space is asserted or needed. $\square$

A.2. The bounded Green function. Define the Green function on the edge (x_n, x_{n+1}) by

$$(91) \quad F_n = -(\mathcal{L}_n)_{11}(x_n, x_{n+1}) - R_n.$$

Using (89) and (19), the same function has the alternative representation

$$(92) \quad F_n = (\mathcal{L}_{n-1})_{22}(x_{n-1}, x_n) + \frac{b_{n-1}^2}{R_{n-1}}.$$

Consequently,

$$(93) \quad (\mathcal{L}_{n-1})_{22} < F_n < -(\mathcal{L}_n)_{11}.$$

The generating functions are smooth on compact parameter tori, so their second derivatives are bounded. Therefore F is a bounded Borel function on $\mathcal{P}_0 \sqcup \mathcal{P}_1$.

More explicitly, set

$$M = \max_{\epsilon \in \{0,1\}} \{ \|(L_\epsilon)_{11}\|_\infty, \|(L_\epsilon)_{22}\|_\infty \},$$

where the norms are taken on the compact parameter tori. Then $|F| \leq M$. Since both curves are parametrized by Euclidean arclength, the physical densities satisfy

$$0 < (L_\epsilon)_{12} \leq 1.$$

Hence

$$\mu(\mathcal{P}_0 \sqcup \mathcal{P}_1) \leq 2\ell_Q \ell_P < \infty.$$

In particular $F \in L^1(\mu)$, and measure invariance gives

$$\int_{\mathcal{P}_0 \sqcup \mathcal{P}_1} (F \circ \mathcal{T} - F) d\mu = 0.$$

This argument requires neither ergodicity nor boundary values of the Green field.

A.3. Recovery of the Hopf inequality.

PROPOSITION A.2 (Green recovery). *The strict second-variation hypothesis implies (25).*

Proof. Using (92) for F_{n+1} and (91) for F_n gives

$$(94) \quad F_{n+1} - F_n = (\mathcal{L}_n)_{11} + (\mathcal{L}_n)_{22} + R_n + \frac{b_n^2}{R_n}.$$

Since $R_n > 0$ and $b_n > 0$, the arithmetic–geometric mean inequality gives

$$R_n + \frac{b_n^2}{R_n} \geq 2b_n.$$

Therefore

$$(95) \quad F_{n+1} - F_n \geq (\mathcal{L}_n)_{11} + 2(\mathcal{L}_n)_{12} + (\mathcal{L}_n)_{22}.$$

Integrate over $\mathcal{P}_0 \sqcup \mathcal{P}_1$. The integral of the left-hand side is zero because F is bounded and the alternating map preserves the finite measure in Lemma 2.3. This proves (25).

The change of variables in the proof of Proposition 3.2 then gives (27). $\square$

Appendix B. The gauge of a translated disk

The translation allowed by the pair theorem has an explicit gauge formula. It also shows directly how a linear odd part can survive without the self condition.

PROPOSITION B.1 (Gauge of a translated disk). *Let $K = D(c, r)$ with $|c| < r$. Then*

$$(96) \quad N_K(x) = \frac{\sqrt{(c \cdot x)^2 + (r^2 - |c|^2)|x|^2} - c \cdot x}{r^2 - |c|^2}.$$

In particular, its odd part is exactly linear:

$$(97) \quad \frac{N_K(x) - N_K(-x)}{2} = -\frac{c \cdot x}{r^2 - |c|^2}.$$

Proof. The boundary condition $x \in t\partial D(c, r)$ is

$$|x - tc| = tr.$$

Squaring and collecting powers of t gives

$$(98) \quad (r^2 - |c|^2)t^2 + 2(c \cdot x)t - |x|^2 = 0.$$

Since $r^2 - |c|^2 > 0$, the unique nonnegative root is precisely the right-hand side of (96). Replacing x by $-x$ and subtracting gives (97). $\square$

The square-root term in (96) is even, and the odd term is the linear functional

$$\lambda(x) = -\frac{c \cdot x}{r^2 - |c|^2}.$$

Along a finite billiard configuration its contribution is $\lambda(q_m) - \lambda(q_0)$. It is therefore an exact endpoint term, not an additional local obstruction to the reflection law. In the self-Minkowski argument, the common-shape conclusion for the table and its polar removes the translation by Lemma 8.1.

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ROBERT JENKINSON ALVAREZ
E-mail: robert.j.jenkinson.alvarez@bath.edu
<https://r02aj.github.io>